Math 419/819 Homework Assignment 1 SEPTEMBER 17, 2003

DUE DATE: SEPTEMBER 24, 2003

The purpose of these assignments is to become familiar with the ideas and definitions from class by doing some small calculations. In answering the questions, there is no need to go over the top with detailed explanations, or to show every single step. You only need to show enough to make it clear that you know what you're doing.

This assignment should take no more than two pages to answer.

1. Let M be the 2-sphere $\{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. The purpose of this problem is to find coordinate charts for M, just slightly different from the ones in class.

Let V_1 be the plane $\{(u, v, 0)\} \subset \mathbb{R}^3$, and V_2 the plane $\{(s, t, 0)\} \subset \mathbb{R}^3$ (we'll consider these two planes to be different for the purpose of providing coordinate charts).

Let U_1 be $M \setminus \{(0, 0, 1)\}$, and define a map $\phi_1 : U_1 \longrightarrow V_1$, by projection from $(0, 0, 1)$. (Projection from a point is the procedure we used in class: for any point p of U_1, look at the line L connecting $(0, 0, 1)$ to p, and define $\phi_1(p)$ to be the point of intersection of the line L with the plane V_1).

Similarly, let U_2 be $M \setminus \{(0, 0, -1)\}$ and define a map $\phi_2 : U_2 \longrightarrow V_2$ by projection from $(0, 0, -1)$.

Work out the formulas for ϕ_1 and ϕ_2, and check that these charts give M the structure of a manifold. (The functions should be similar to the ones from class. The only difference should be some multiples of 2).

2. For what exponents r ($r \in \mathbb{R}$) is the function $(z - 1)^r$ on $\mathbb{R}^3$ a C^∞ function on the manifold M from question 1? (Include a brief explanation for your answer.)

3. Let $\mathbb{P}^1$ be the Riemann sphere again. We can, as in class, describe a holomorphic map $\pi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ from $\mathbb{P}^1$ to itself by defining it as $z \mapsto z^2$ and $w \mapsto w^2$ on each of the two coordinate charts. (We will check in class that these match up).

Using our first description of $\mathbb{P}^1$ as the subset

$$\{(a, b, c) \mid a^2 + b^2 + c^2 = 1\} \subset \mathbb{R}^3,$$

it should be possible to describe the map π using (a, b, c) coordinates.

Show that in these coordinates the map π is given by the rather awkward rule:

$$(a, b, c) \mapsto \left(\frac{2(a^2 - b^2)(1 - c^2)}{(1 - c^2)^2 + (1 - c)^4}, \frac{4ab(1 - c^2)}{(1 - c^2)^2 + (1 - c)^4}, \frac{(1 - c^2)^2 - (1 - c)^4}{(1 - c^2)^2 + (1 - c)^4} \right).$$

This is more evidence for the fact that it's usually easier to describe the map in charts than to give a global description.