The answers to the following problems should again be short – no long essay is required!

1. Several times in class I've mentioned that the only functions $\mathbb{P}^1 \longrightarrow \mathbb{C}$ which are holomorphic in all charts (a *global holomorphic function*) are the constant functions. The purpose of this question is to see why this is true.

Suppose that $f : \mathbb{P}^1 \longrightarrow \mathbb{C}$ is a global holomorphic function. In terms of the chart V_1, that means we should get a holomorphic function $f_1 : V_1 \longrightarrow \mathbb{C}$, which will therefore have a power series expansion:

$$f_1(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots\cdots$$

the expansion has infinite radius of convergence since the function is defined on all of $V_1 \cong \mathbb{C}$.

Similarly, in terms of the chart V_2, we get a holomorphic function $f_2 : V_2 \longrightarrow \mathbb{C}$, with power series expansion

$$f_2(w) = b_0 + b_1 w + b_2 w^2 + b_3 w^3 + b_4 w^4 + \cdots\cdots$$

On the overlap, the charts are related by the transition function $w = 1/z$ (or, $z = 1/w$). Since the functions $f_1(z)$ and $f_2(w)$ must match up on the intersection of the two charts, explain why this forces both f_1 and f_2 (and hence f) to be constant.

The purpose of the next two questions is to think about the basic details of ramified covers, and to practice using the Riemann-Hurwitz formula.

2. Suppose that $\pi : X_1 \longrightarrow X_2$ is a map of degree d between Riemann surfaces.

(a) For any point p of X_1, explain why the ramification index k_p cannot be bigger than d, i.e., that $k_p \leq d$.

(b) If $d = 2$ (a *double cover*) explain why a point p of X_1 is either not a ramification point, or has ramification index exactly 2.

(c) Again in the case that $d = 2$ explain why the number of branch points (on X_2) is the same as the number of ramification points (on X_1).

Parts (b) and (c) show that for degree 2 covers, the topological data (the number and type of ramification points) of a map $\pi : X_1 \longrightarrow X_2$ is given just by knowing the number of ramification or branch points. Use this to answer the following two questions.

(d) Suppose that $\pi : X_1 \longrightarrow X_2$ is a degree 2 map. Show that the number of ramification points is *even.*

(e) Suppose that $\pi : X_1 \longrightarrow \mathbb{P}^1$ is a double cover, with $2t$ branch points. What is the genus of X_1?.

3. Use the Riemann-Hurwitz formula to find the genus of X_1, the genus of X_2, or the number of ramification points, as required.

(a) $\pi : X_1 \longrightarrow \mathbb{P}^1$ is a degree 3 cover, with two ramification points, both with ramification index $k_p = 3$. Find the genus of X_1.

(b) $\pi : X_1 \longrightarrow \mathbb{P}^1$ is a degree 3 cover, with three ramification points, all with ramification index $k_p = 3$. Find the genus of X_1.

(c) $\pi : X_1 \longrightarrow X_2$ is a map of degree d, X_1 has genus 1, and there are no ramification points. Find the genus of X_2.

(d) X_1 is of genus g, X_2 is of genus 1, the map $\pi : X_1 \longrightarrow X_2$ is of degree d, and all ramification points p in X_1 are of index 2. Find the number of ramification points (the answer turns out, in this case, not to depend on the degree d).

Can you think of a map $X_1 \longrightarrow \mathbb{P}^1$ satisfying the description in part (a)?