Math 419/819 Homework Assignment 3 October 21, 2003

due date: October 28, 2003

Notational Conventions: The symbol F will always mean a homogeneous polynomial in X, Y and Z. A symbol f will mean a non-homogeneous polynomial. The symbols F_X, F_Y, and F_Z will denote the partial derivatives of F with respect to X, Y and Z. Similarly, f_x and f_y are the partial derivatives of f.

1.

(a) For the homogeneous polynomial $F = X^3Y^4 - X^5YZ + 5X^3Z^4 + 3Y^7$, find the corresponding equations in charts 1, 2, and 3.

(b) For the polynomial $f = x_1^4 - x_1^2y_1^3 + 3xy - 2$ in chart 1, find the corresponding homogeneous polynomial in $\mathbb{P}^2$. What are the polynomials in charts 2 and 3 corresponding to this homogeneous polynomial?

(c) Check that the zero loci of the polynomials f_1, f_2, and f_3 from part (b) patch together. That is, show that on the intersection of the charts the corresponding two polynomials differ (multiplicatively) by a function which has no zero on the intersection of the charts.

Note: I realize that our arguments about homogeneous polynomials make part (c) automatic, but doing this calculation now will help when we come to “line bundles” later.

2. Prove the following lemma, first noticed by Euler. If F is a homogeneous polynomial of degree d, then

$$X \cdot F_X + Y \cdot F_Y + Z \cdot F_Z = d \cdot F$$

(Hint: Use linearity to reduce to the case that F is a monomial.)

3. Suppose that $f(x, y)$ is a polynomial (or analytic function) on $\mathbb{C}^2$ such that $C \leftrightarrow f = 0$ is a smooth submanifold. (i.e., there are no points p in $\mathbb{C}^2$ with $f(p) = 0$, $f_x(p) = 0$ and $f_y(p) = 0$.)

(a) Assume that $p = (0, 0)$ is on C, or equivalently, that near $(0, 0)$ f looks like

$$f(x, y) = 0 + a_{(1,0)}x + a_{(0,1)}y + a_{(2,0)}x^2 + a_{(1,1)}xy + a_{(0,2)}y^2 + \cdots$$

for coefficients $a_{(i,j)}$. Show that the tangent line to C at $p = (0, 0)$ is given by

$$a_{(1,0)}x + a_{(0,1)}y = 0.$$

(b) Suppose that $p = (p_x, p_y)$ is any point of C, and show that the tangent line to C at p is given by

$$f_x(p)x + f_y(p)y - (f_x(p)p_x + f_y(p)p_y) = 0.$$

4. Let F be a homogeneous polynomial. We say that F defines a submanifold of $\mathbb{P}^2$ if in each of the three charts the corresponding function $f_i = 0$ defines a submanifold (as described in question 3). Show that this condition is the same as the condition that F_X, F_Y and F_Z have no common solution other than $P = (0, 0, 0)$ (i.e., they have no common point in $\mathbb{P}^2$). (HINT: First consider only chart 1, and the corresponding nonhomogeneous polynomial f. Use problem 2 to show that for chart 1 the condition that $f_x(p)$, $f_y(p)$ and $f(p)$ have no common solutions is the same as $F_X(P)$, $F_Y(P)$, and $F_Z(P)$ having no common solutions. Then use symmetry to argue that this is the correct condition for charts 2 and 3 as well.)

5. Suppose that F is a homogeneous polynomial such that $C \leftrightarrow F = 0$ is a smooth submanifold of $\mathbb{P}^2$, as in question 4. Show that the tangent line to a point p of C is given by

$$F_X(p)X + F_Y(p)Y + F_Z(p)Z = 0$$

in $\mathbb{P}^2$. (SUGGESTION: Use problem 2 and part(b) of problem 3.)

For definition of tangent line, you can take the following: Look in any chart of $\mathbb{P}^2$ containing p, and work out the tangent line L there. The tangent line in $\mathbb{P}^2$ is just the corresponding line in $\mathbb{P}^2$.