Math 419/819 Homework Assignment 4 NOVEMBER 21, 2003

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1. Suppose that F is a homogeneous polynomial of degree d in X, Y, and Z. If we make the substitution $Z = tX$, we get a polynomial

$$F_t = \sum_{j=0}^{d} a_j(t) X^j Y^{d-j}$$

whose coefficients $a_j(t)$ are polynomials in t. Explain why $\deg(a_j) \leq j$, and also explain which terms of F contribute to the top coefficient of t in $a_j(t)$.

2. Suppose that we fix a value t_0 of t. Explain how a solution (in X and Y) to the equation $F_{t_0} = 0$ gives you a point on $F = 0$ in $\mathbb{P}^2$. How to you find the Z coordinate of this point?

3. Suppose that G is a homogeneous polynomial of degree e in X, Y, Z, and

$$G_t = \sum_{j=0}^{e} b_j(t) X^j Y^{e-j}$$

the polynomial resulting from the substitution $Z = tX$. For a fixed value t_0 of t, explain why finding a point p in the intersection $\{F = 0\} \cap \{G = 0\}$ in $\mathbb{P}^2$ such that p is also on the line $Z = t_0 X$ is the same as finding a common solution (in X and Y) to the equations $F_{t_0} = 0$ and $G_{t_0} = 0$.

4. Suppose that no point of $\{F = 0\} \cap \{G = 0\}$ lies on the line $X = 0$. Explain why all points of the intersection have to lie on some line of the form $Z = t_0 X$, for some t_0. Suppose further that no two points of the intersection $\{F = 0\} \cap \{G = 0\}$ lie on the *same* line of this form. Explain why points of the intersection are in one to one correspondence with the values of t such that the determinant

$$R(t) := \begin{vmatrix} a_d(t) & a_{d-1}(t) & a_{d-2}(t) & \cdots & a_1(t) & a_0(t) & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_d(t) & a_{d-1}(t) & \cdots & a_2(t) & a_1(t) & a_0(t) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & a_1(t) & a_0(t) \\ b_e(t) & b_{e-1}(t) & b_{e-2}(t) & \cdots & b_0(t) & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_e(t) & b_{e-1}(t) & \cdots & b_1(t) & b_0(t) & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & b_1(t) & b_0(t) \end{vmatrix}$$

of the $(d+e) \times (d+e)$ matrix above is zero.

5. Using the bounds on $\deg(a_j)$ (and of course $\deg(b_j)$) from question 1, show that $\deg(R(t)) \leq d \cdot e$. The formula

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) M_{1\sigma(1)} M_{2\sigma(2)} M_{3\sigma(3)} \cdots M_{n-1\sigma(n-1)} M_{n\sigma(n)}$$

for the determinant of an $n \times n$ matrix M might be useful. Here S_n is the permutation group on n symbols, and $\operatorname{sgn}(\sigma)$ of a permutation σ is ± 1 depending on whether σ is an even or odd permutation. (HINT: Extend the bound on degree to the zero entries of the matrix in question 4 so as to give you a pattern which is easy to analyze. Since the entries are zero, any value of the "degree" is permitted for these entries, even negative ones – any piece of the formula involving these entries will be zero anyway. Use the pattern to compute the degree in t of any monomial in the sum above.)

6. We now want to show (under our assumptions about F and G) that $R(t)$ must be of degree precisely $d \cdot e$. This won't be true for arbitrary F and G. For instance, for $F = X^2 + 2Y^2 + 18Z^2 + 12YZ$ and $G = X^3 + Y^3 + 27Z^3$, we get the matrix

$$\begin{bmatrix} (18t^2+1) & 12t & 2 & 0 & 0 \\ 0 & (18t^2+1) & 12t & 2 & 0 \\ 0 & 0 & (18t^2+1) & 12t & 2 \\ (27t^3+1) & 0 & 0 & 1 & 0 \\ 0 & (27t^3+1) & 0 & 0 & 1 \end{bmatrix}$$

with determinant $2196t^4 + 54t^2 + 72t + 9$, of degree strictly smaller than 6.

Consider the matrix made up by replacing each entry in the resultant matrix (of the form $a_j(t)$ or $b_j(t)$) by the coefficient of the top possible power of t in that spot (i.e., the coefficient of t^j in each case). In the example above, this would be the matrix

$$\begin{bmatrix} 18 & 12 & 2 & 0 & 0 \\ 0 & 18 & 12 & 2 & 0 \\ 0 & 0 & 18 & 12 & 2 \\ 27 & 0 & 0 & 1 & 0 \\ 0 & 27 & 0 & 0 & 1 \end{bmatrix}.$$

Explain why (no matter what F and G are) that the coefficient of the $d \cdot e$-th power of t in $R(t)$ is the determinant of this matrix, so that $\deg(R(t)) = d \cdot e$ if and only if this determinant is not zero.

7. Considering where the coefficient of t^j in $a_j(t)$ comes from (problem 1) explain why the vanishing of this determinant is the same as saying that the polynomials F' and G' in Y and Z obtained from F and G by setting $X = 0$ have a common root.

8. Explain why this is equivalent to $\{F = 0\} \cap \{G = 0\}$ having a point of intersection on the line $X = 0$. Thus, under the assumptions of question 4, degree $R(t) = d \cdot e$.