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EQUIVARIANT TRIVIALITY THEOREMS FOR HILBERT C^*-MODULES

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ABSTRACT. The purpose of this paper is to give an exposition of the various triviality theorems, the equivariant version of a result due to L. Brown, and a simplification of the proof of Kasparov's triviality theorems.

0. Introduction and notation. In [**5**] several triviality theorems are given for continuous fields of Hilbert spaces $(\mathcal{H}(z), \Gamma)$ over a paracompact space B. When B is locally compact and $\mathcal{E}$ is the subspace of Γ of functions vanishing at infinity, then $\mathcal{E}$ is a Hilbert $C_0(B)$-module.

Recently some of these triviality theorems [**5**, Théorème 4 and Corollaire 3] have been generalized to the case of Hilbert C^*-modules for noncommutative algebras [**2**, **6**, **7**, **9**]. Our purpose is to give an exposition of the various triviality theorems, the equivariant version of the triviality theorem of [**2**], and a simplification of the proof of Kasparov's triviality theorems [**7**, **9**].

Although Hilbert C^*-modules had been considered earlier than [**7**] (see e.g. [**10**]), we will adopt the notation of Kasparov [**7**, §2, Definitions 1–4]. If $\mathcal{E}$ is a Hilbert A-module then $\mathcal{E}^\infty$ denotes the direct sum of $\mathcal{E}$ with itself countably many times; an isomorphism of Hilbert A-modules is denoted by $\simeq$. $\mathcal{H}_A$ denotes A^∞ where A is considered a module over itself [**7**, §2, Example 1].

The two triviality theorems then are

THEOREM 1.4 [**5**, **6**, **7**, **9**]. *Let $\mathcal{E}$ be a countably generated Hilbert A-module; then* $\mathcal{E} \oplus \mathcal{H}_A \simeq \mathcal{H}_A$.

THEOREM 1.9 [**2**, **5**, **6**]. *Let $\mathcal{E}$ be a full countably generated Hilbert A-module. If A has a strictly positive element, then $\mathcal{E}^\infty \simeq \mathcal{H}_A$.*

1. Triviality theorems without group actions. In this section we consider the triviality theorems mentioned in §0 but without any group actions. A crucial notion in this section is that of a strictly positive element.

DEFINITION 1.1 [**1**]. If e is a positive element of a C^*-algebra A and $\phi(e) \neq 0$ for all states ϕ on A, then e is *strictly positive*.

The following lemma, observed in [**2**], can be deduced from [**1**], but since it has a straightforward proof, we give it here.

LEMMA 1.2. *If e is a positive element of A then e is strictly positive if and only if eA is dense in A.*

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PROOF. Suppose eA is not dense in A. Then by [**4**, 2.9.4] there is a state of A vanishing on eA. Such a state must vanish on e, so e is not strictly positive.

Suppose ϕ is a state of A for which $\phi(e) = 0$. Then, by the Cauchy-Schwarz inequality, ϕ vanishes on eA. Thus eA is not dense. Q.E.D.

Now we will apply Lemma 1.2 to the algebra $\mathcal{K}(\mathcal{E})$ (see [**7**, §2, Definition 4]).

LEMMA 1.3. *If $\mathcal{E}$ is a Hilbert A-module and T is a positive element of $\mathcal{K}(\mathcal{E})$, then T is strictly positive if and only if T has dense range.*

PROOF. If T is strictly positive then $T\mathcal{K}(\mathcal{E})$ is dense in $\mathcal{K}(\mathcal{E})$. As $\overline{\mathcal{K}(\mathcal{E})\mathcal{E}} = \mathcal{E}$, we have $\overline{T\mathcal{E}} = \overline{T\mathcal{K}(\mathcal{E})\mathcal{E}} = \overline{\mathcal{K}(\mathcal{E})\mathcal{E}} = \mathcal{E}$.

If T has dense range, then given $\xi \in \mathcal{E}$ there exists a sequence $\xi_n \in \mathcal{E}$ such that $T\xi_n \to \xi$. So $\theta_{\xi,\eta} = \lim_n T\theta_{\xi_n,\eta} \in \overline{T\mathcal{K}(\mathcal{E})}$. So $T\mathcal{K}(\mathcal{E})$ is dense and T is strictly positive. Q.E.D.

Next we give a proof of the stabilization theorem. The original version of this theorem, for continuous fields of Hilbert spaces, is Théorème 4 (p. 259) of [**5**]. A C^*-algebra version is given in Theorem 3.1 of [**2**]. In this version B is a hereditary C^*-subalgebra of A with strictly positive element, $\mathcal{E} = \overline{BA}$, $D = \mathcal{K}(\mathcal{E} \oplus A^\infty)$, and $p \in M(D)$ is the projection onto $\mathcal{E}$. Then $1 - p \sim 1$ means $A^\infty \simeq \mathcal{E} \oplus A^\infty$.

The proofs of [**6** and **7**] follow a Gram-Schmidt orthogonalization procedure. The proof below, using polar decomposition, is perhaps simpler.

THEOREM 1.4 (STABILIZATION). *If $\mathcal{E}$ is a countably generated Hilbert A-module, then $\mathcal{E} \oplus \mathcal{H}_A \simeq \mathcal{H}_A$.*

PROOF. We may assume A is unital; in fact, $\mathcal{E}$ may be considered an $\tilde{A}$-module; then $\mathcal{E} \oplus \mathcal{H}_{\tilde{A}} \simeq \mathcal{H}_{\tilde{A}}$ implies $\mathcal{E} \oplus \mathcal{H}_A \cong \mathcal{H}_A$, as $\overline{\mathcal{E}A} = \mathcal{E}$ and $\overline{\mathcal{H}_{\tilde{A}}A} = \mathcal{H}_A$.

Let $\{\eta_i\}_{i=1}^\infty \subseteq E$ be a bounded countable set of generators with each generator repeated infinitely often. Let $\{\xi_i\} \subseteq \mathcal{H}_A$ be the standard orthnormal basis; that is, ξ_i is the sequence with zeros everywhere but the ith place, where there is a 1. Define T: $\mathcal{H}_A \to \mathcal{E} \oplus \mathcal{H}_A$ by

$$T(\xi_i) = 2^{-i}\eta_i \oplus 4^{-i}\xi_i.$$

It is clear that $T \in \mathcal{L}(\mathcal{H}_A, \mathcal{E} \oplus \mathcal{H}_A)$; in fact,

$$T = \sum 2^{-i}\theta_{\eta_i \oplus 2^{-i}\xi_i, \xi_i} \in \mathcal{K}(\mathcal{H}_A, \mathcal{E} \oplus \mathcal{H}_A).$$

As each η_i is repeated infinitely often, $\eta_i \oplus 2^{-k}\xi_k \in \operatorname{ran}(T)$ for infinitely many k's. So $\eta_i \oplus 0 \in \operatorname{ran}(T)^-$ and thus $0 \oplus \xi_i \in \operatorname{ran}(T)^-$; thus $\operatorname{ran}(T)$ is dense in $\mathcal{E} \oplus \mathcal{H}_A$. Now

$$T^*T = \begin{pmatrix} 4^{-4} & & & 0 \\ & 4^{-8} & & \\ & & 4^{-12} & \\ 0 & & & \ddots \end{pmatrix} + \begin{pmatrix} 4^{-2}\langle \eta_1, \eta_1 \rangle & 4^{-3}\langle \eta_1, \eta_2 \rangle & \cdots \\ 4^{-3}\langle \eta_2, \eta_1 \rangle & 4^{-4}\langle \eta_2, \eta_2 \rangle & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

$$= K + K^1 \quad \text{with } K, K^1 \geqslant 0.$$

It is clear that ran(K) is dense so K is strictly positive. Thus T^*T is strictly positive. So ran(T^*T) is dense and thus ran($|T|$) is also dense. Finally, define $V\colon \mathcal{H}_A \to \mathcal{E} \oplus \mathcal{H}_A$ by $V(|T|\xi) = T\xi$. As $\|V(|T|\xi)\| = \||T|\xi\|$, V has a continuous extension to $\mathcal{H}_A$, where it becomes a unitary from $\mathcal{H}_A$ to $\mathcal{E} \oplus \mathcal{H}_A$. Q.E.D.

COROLLARY 1.5. *If $\mathcal{E}$ is a Hilbert A-module then $\mathcal{E}$ is countably generated if and only if $\mathcal{K}(\mathcal{E})$ has a strictly positive element.*

PROOF. As in the proof of Theorem 1.4 we may suppose A is unital. By Theorem 1.4 there is a projection P in $\mathcal{L}(\mathcal{H}_A)$ with $\mathcal{E} \cong P(\mathcal{H}_A)$. Let $\{\xi_n\}$ be the standard orthonormal basis for $\mathcal{H}_A$. Then $K = \sum 1/n\theta_{\xi_n,\xi_n}$ is a strictly positive element of $\mathcal{K}(\mathcal{H}_A)$ by Lemma 1.3. Now $\mathcal{K}(\mathcal{E}) \cong P\mathcal{K}(\mathcal{H}_A)P$, so $\mathcal{K}(\mathcal{E})$ has a strictly positive element [**3**, Proposition 2.3], PKP.

Conversely, if $\mathcal{K}(\mathcal{E})$ has strictly positive element $K = \sum_{i=1}^{\infty}\theta_{\xi_i,\eta_i}$ with $\xi_i, \eta_i \in \mathcal{E}$, then as $K\mathcal{E}$ is dense, $\{\xi_i\}_{i=1}^{\infty}$ is a set of generators.

DEFINITION 1.6. *If $\mathcal{E}$ is a Hilbert A-module then $\langle \mathcal{E}, \mathcal{E}\rangle = \{\sum\langle \xi_i, \eta_i\rangle\colon \xi_i, \eta_i \in \mathcal{E}\}^-$ is called the* support *of $\mathcal{E}$. $\mathcal{E}$ is* full *if $\langle \mathcal{E}, \mathcal{E}\rangle = A$.*

LEMMA 1.7. *If $\mathcal{E}$ is a full Hilbert A-module and A has a strictly positive element then there is a sequence $\{\xi_i\}$ in $\mathcal{E}$ such that $\sum\langle \xi_i, \xi_i\rangle = 1$ strictly in $M(A)$.*

PROOF. This is precisely the statement of Lemma 2.3 of [**2**] when $\mathcal{E} = pA$ and $\langle \xi, \eta\rangle = \xi^*\eta$ for p a projection in $M(A)$ and $\xi, \eta \in \mathcal{E}$. The proof goes over to the more general case with obvious modifications. Q.E.D.

COROLLARY 1.8. *If $\mathcal{E}$ is a full Hilbert A-module and A has strictly positive element then $\mathcal{E}^\infty \simeq A \oplus \mathcal{F}$ for some Hilbert A-module $\mathcal{F}$.*

PROOF. Let $\{\xi_i\}$ be as in Lemma 1.7. Define $T\colon A \to \mathcal{E}^\infty$ by $T(a) = (\xi_i a)$. As $\langle(\xi_i a), (\xi_i a)\rangle = a^*a$, we see that $(\xi_i a) \in \mathcal{E}^\infty$. Define $T^*\colon \mathcal{E}^\infty \to A$ by $T^*(\eta_i) = \sum\langle \xi_i, \eta_i\rangle$. By applying the Cauchy-Schwarz inequality we see that $\sum\langle \xi_i, \eta_i\rangle$ converges in norm to an element of A. As $T^*T = \mathrm{id}_A$ we have that $T \oplus \mathrm{id}\colon A \oplus (1 - TT^*)\mathcal{E}^\infty \to \mathcal{E}^\infty$ is an isomorphism. Q.E.D.

THEOREM 1.9. *If $\mathcal{E}$ is a countably generated full Hilbert A-module and A has a strictly positive element, then $\mathcal{E}^\infty \simeq \mathcal{H}_A$.*

PROOF. $\mathcal{E}^\infty \simeq (A \oplus \mathcal{F})^\infty = \mathcal{H}_A \oplus \mathcal{F}^\infty \simeq \mathcal{H}_A$, where the last isomorphism follows from the stabilization theorem because $\mathcal{F}$, being a complemented submodule of $\mathcal{E}^\infty$, is countably generated. Q.E.D.

REMARK 1.10. With Theorem 1.9 we may quickly obtain a proof of [**3**, Theorem 1.2]. Suppose A and B are strongly Morita equivalent; in our notation this means that there is a full Hilbert B-module $\mathcal{E}$ with $A \cong \mathcal{K}(\mathcal{E})$. If A and B have strictly positive elements then $\mathcal{E}$ is countably generated by Corollary 1.5 and we may apply Theorem 1.9 to conclude that $\mathcal{E}^\infty \cong \mathcal{B}^\infty$. Now, as in [**8**, §2.9],

$$\mathcal{K}(\mathcal{E}^\infty) = \mathcal{K}(\mathcal{E} \otimes \mathcal{H}) \cong \mathcal{K}(\mathcal{E}) \otimes \mathcal{K}(\mathcal{H})$$

and, similarly, $\mathcal{K}(\mathcal{B}^\infty) \cong \mathcal{K}(\mathcal{B}) \otimes \mathcal{K}(\mathcal{H})$. Thus

$$A \otimes \mathcal{K} \cong \mathcal{K}(\mathcal{E}) \otimes \mathcal{K}(\mathcal{H}) \cong \mathcal{K}(\mathcal{E}^\infty) \cong \mathcal{K}(\mathcal{B}^\infty) \cong \mathcal{K}(\mathcal{B}) \otimes \mathcal{K}(\mathcal{H}) \cong B \otimes \mathcal{K}.$$

So A and B are stably isomorphic.

2. Triviality theorems with group actions. Let (A, α, G) be a C^*-dynamical system.

DEFINITION 2.1 (SEE [**7**, DEFINITION 1]). A *Hilbert* $(G - A)$*-module* $\mathcal{E}$ is a Hilbert A-module which is also a left G-module satisfying:

(i) $t \cdot (\xi a) = (t \cdot \xi)\alpha_t(a)$,

(ii) $t \to t \cdot \xi$ is continuous,

(iii) $\langle t \cdot \xi, t \cdot \eta \rangle = \alpha_t(\langle \xi, \eta \rangle)$

for all $\xi, \eta \in \mathcal{E}$, $t \in G$ and $a \in A$.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert $(G - A)$-modules. There is an action of G induced on $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$, namely $(t \cdot T)(\xi) = t \cdot T(t^{-1} \cdot \xi)$ for $\xi \in \mathcal{E}_1$, $T \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ and $t \in G$. Note that T is G-equivariant iff $t \cdot T = T$ for all $t \in G$. In general, the map $t \to t \cdot T$ is strongly continuous. T is called *G-continuous* in case this map is continuous in norm (see [**9**, 1.3]).

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are Hilbert $(G - A)$-modules then we can make $\mathcal{E}_1 \oplus \mathcal{E}_2$ into a Hilbert $(G - A)$-module by defining the G action as follows: $t \cdot (\xi_1, \xi_2) = (t \cdot \xi_1, t \cdot \xi_2)$ for $t \in G$, $\xi_1 \in \mathcal{E}_1$, and $\xi_2 \in \mathcal{E}_2$. Similarly, if $\mathcal{E}$ is a Hilbert $(G - A)$-module then so is $\mathcal{E}^\infty$. A itself is a Hilbert $(G - A)$-module where $t \cdot \xi = \alpha_t(\xi)$ for $t \in G$ and $\xi \in A$.

If $\mathcal{E}$ is a Hilbert $(G - A)$-module we can make $C_{00}(G, \mathcal{E})$ (the continuous compactly supported functions from G to $\mathcal{E}$) into a pre-Hilbert $(G - A)$-module as follows:

$$(\xi a)(t) = \xi(t)a, \quad (s \cdot \xi)(t) = s \cdot \xi(s^{-1}t), \quad \langle \xi, \eta \rangle = \int_G \langle \xi(t), \eta(t) \rangle \, dt$$

for $\xi, \eta \in C_{00}(G, \mathcal{E})$, $s \in G$ and $a \in A$.

DEFINITION 2.2 (SEE [**9**, 1.4]). $L^2(G, \mathcal{E})$ is the completion of $C_{00}(G, \mathcal{E})$ as a Hilbert $(G - A)$-module.

Note that $L^2(G, \mathcal{E})$ is a completion of the algebraic tensor product $L^2(G) \otimes \mathcal{E}$ and the G action is the tensor product of the left regular representation with the G action on $\mathcal{E}$. In view of [**4**, 13.11.3], the following result should not be surprising.

LEMMA 2.3. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic as Hilbert A-modules then $L^2(G, \mathcal{E}_1)$ and $L^2(G, \mathcal{E}_2)$ are isomorphic as Hilbert $(G - A)$-modules (i.e. by a G-equivariant isomorphism of A-modules).*

PROOF. Let U be a unitary operator in $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$. Define $V \in \mathcal{L}(L^2(G, \mathcal{E}_1), L^2(G, \mathcal{E}_2))$ by $(V\xi)(t) = t \cdot U(t^{-1} \cdot \xi(t))$ for $\xi \in C_{00}(G, \mathcal{E}_1)$. It is not difficult to check that V is an A-module map, G-equivariant and unitary. Q.E.D.

The Hilbert $(G - A)$-module version of Theorem 1.9 now follows.

THEOREM 2.4. *Let $\mathcal{E}$ be a Hilbert $(G - A)$-module which is countably generated and full as a Hilbert A-module. Then $L^2(G, \mathcal{E})^\infty$ is isomorphic to $L^2(G, A)^\infty$ by a G-equivariant isomorphism of Hilbert A-modules.*

PROOF. There are obvious G-equivariant isomorphisms $L^2(G, \mathcal{E})^\infty \simeq L^2(G, \mathcal{E}^\infty)$ and $L^2(G, A)^\infty \simeq L^2(G, A^\infty)$. By Theorem 1.9 $\mathcal{E}^\infty$ and A^∞ are isomorphic as Hilbert A-modules and so by Lemma 2.3 there is a G-equivariant isomorphism $L^2(G, \mathcal{E}^\infty) \simeq L^2(G, A^\infty)$. Q.E.D.

The Hilbert $(G - A)$-module version of Theorem 1.4 is the following:

THEOREM 2.5 (KASPAROV [9, THEOREM 2.1]). *Let $\mathcal{E}$ be a Hilbert $(G - A)$-module which is countably generated as a Hilbert A-module. There is a G-continuous isomorphism from $\mathcal{E} \oplus L^2(G, A)^\infty$ to $L^2(G, A)^\infty$. If G is compact this isomorphism can be chosen to be G-equivariant.*

PROOF. By Lemma 2.3 and Theorem 1.4 we have equivariant isomorphisms

$$L^2(G, A)^\infty \simeq L^2(G, A^\infty)^\infty \simeq L^2(G, \mathcal{E} \oplus A^\infty)^\infty \simeq L^2(G, \mathcal{E})^\infty \oplus L^2(G, A^\infty)^\infty.$$

Let $\phi \in C_{00}(G)$ with $\|\phi\|_2 = 1$. Let $V\colon \mathcal{E} \to L^2(G, \mathcal{E})$ be given by $(V\xi)(t) = \xi\phi(t)$. It is easy to check that V is a G-continuous isometry. Now define $U\colon \mathcal{E} \oplus L^2(G, \mathcal{E})^\infty \to L^2(G, \mathcal{E})^\infty$ by

$$U(\xi_0, \xi_1, \xi_2, \ldots) = (V\xi_0 + (1 - VV^*)\xi_1, VV^*\xi_1 + (1 - VV^*)\xi_2, \ldots).$$

U defines a G-continuous unitary with

$$U^*(\eta_1, \eta_2, \eta_3, \ldots) = (V^*\eta_1, VV^*\eta_2 + (1 - VV^*)\eta_1, VV^*\eta_3 + (1 - VV^*)\eta_2, \ldots).$$

Thus

$$\begin{aligned}\mathcal{E} \oplus L^2(G, A)^\infty &\simeq \mathcal{E} \oplus L^2(G, \mathcal{E})^\infty \oplus L^2(G, A^\infty)^\infty \\ &\simeq L^2(G, \mathcal{E})^\infty \oplus L^2(G, A^\infty)^\infty \simeq L^2(G, A)^\infty.\end{aligned}$$

The resulting isomorphism is G-continuous.

If G is compact we may take $\phi = 1$. Then V and U are equivariant and thus $\mathcal{E} \oplus L^2(G, A)^\infty \simeq L^2(G, A)^\infty$ by a G-equivariant unitary. Q.E.D.

To conclude we shall explain why Theorem 2.4 is the equivariant version of the triviality theorem of [**2**]. The equivariant version of [**2**, Lemma 2.5] is

COROLLARY 2.6. *Let (A, α, G) be a C^*-dynamical system and suppose A has a strictly positive element. If p in $M(A)$ is a full invariant projection then $p \otimes 1 \sim 1 \otimes 1$ in $M(A \otimes \mathcal{K}(L^2(G)^\infty))$ by an invariant partial isometry.*

Theorem 1.9 and Corollary 2.6 (in the case of a trivial action) are, in fact, proving the same thing. Indeed, suppose B has a strictly positive element and $\mathcal{E}$ is a countably generated full Hilbert B-module. Let $A = \mathcal{K}(\mathcal{E} \oplus B)$; A is the linking algebra for the strongly Morita equivalent C^*-algebras $\mathcal{K}(\mathcal{E})$ and B as in [**2**, Theorem 2.8 and **3**, Theorem 1.1]. By Lemma 1.2, B, when considered as a Hilbert B-module, is countably generated (by a single element in fact). Thus $\mathcal{E} \oplus B$ is countably generated as a Hilbert B-module; so by Corollary 1.5, $A = \mathcal{K}(\mathcal{E} \oplus B)$ has a strictly positive element. Let p and q be the projections in $M(A)$ with ranges $\mathcal{E}$ and B, respectively. It is easy to check that AqA is dense in A and similarly ApA is dense in A because $\mathcal{E}$ is full. Thus p and q are full projections [**2**, Lemma 1.1].

Now as in [8, §2.9] $\mathcal{K}(\mathcal{E} \oplus B) \otimes \mathcal{K} \cong \mathcal{K}((\mathcal{E} \oplus B) \otimes \mathcal{K})$, so

$$A \otimes \mathcal{K} \cong \mathcal{K}(\mathcal{E} \otimes \mathcal{K} \oplus B \otimes \mathcal{K}).$$

Under this isomorphism $p \otimes 1$ and $q \otimes 1$ become the projections onto $\mathcal{E} \otimes \mathcal{K} \cong \mathcal{E}^\infty$ and $B \otimes \mathcal{K} \cong B^\infty$, respectively. Thus $p \otimes 1 \sim 1 \otimes 1 \sim q \otimes 1$ gives $\mathcal{E}^\infty \cong B^\infty$.

PROOF OF COROLLARY 2.6. Let $\mathcal{E} = pA$; then $\mathcal{E}$ is a Hilbert $(G - A)$-module. As $\mathcal{K}(\mathcal{E}) = pAp$, which being a corner of A has a strictly positive element [3, Proposition 2.3], we have that $\mathcal{E}$ is countably generated by Corollary 1.5. Also, as $\langle \mathcal{E}, \mathcal{E} \rangle = \overline{pAp}$ we see that $\mathcal{E}$ is full. So $\mathcal{E} \otimes L^2(G)^\infty \cong A \otimes L^2(G)^\infty$ by an equivariant isomorphism, that is, $p \otimes 1 \sim 1 \otimes 1$ in

$$\mathcal{L}\left(A \otimes L^2(G)^\infty\right) \cong M\left(A \otimes \mathcal{K}\left(L^2(G)^\infty\right)\right)$$

by an invariant partial isometry. Q.E.D.

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