Seminar on Brown Measure · 16 April 2021

- (M, τ) finite von Neumann algebra, τ faithful normal trace, $\tau(1) = 1$
- $a \in M$, $|a| = \sqrt{a^*a}$, $\mu_{|a|}$ = spectral measure of $|a|$: $\tau(f(|a|)) = \int_{\mathbb{R}} f(t)\, d\mu_{|a|}(t)$
- $L(a) = \int \log(t)\, d\mu_{|a|}(t) = \tau(\log |a|) \in [-\infty, \infty)$
- $L(a^*a) = \frac{1}{2} L(a)$
- $0 \leq a \leq b \Rightarrow L(a) \leq L(b)$
- $L_\varepsilon(a) = \frac{1}{2} L(a^*a + \varepsilon)$, $L(a) = \inf_{\varepsilon > 0} L_\varepsilon(a)$
- L is upper semi-continuous (usc)
- $L(z1) = \log|z|$
- $L(0) = -\infty$
- $L(u) = 0$ if $u^*u = uu^* = 1$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $\cdot\ a \in M^{-1} \Rightarrow L(a) > -\infty$

- if $d\mu_{|a|}(t) = f(t)\,dt$ with $f \in \bigcup_{1 < p \leq \infty} L^p(\mathbb{R})$ then $L(a) > -\infty$

Goal for today: $L(ab) = L(a) + L(b)$ (2)

Def: $I \subseteq \mathbb{R}$ open interval $a : I \to M$

a is C^1 if $\exists\, b : I \to M$ $\begin{bmatrix} \text{norm} \\ \text{cont} \end{bmatrix}$

st $\forall t_0$ $\quad a(t) - a(t_0) = b(t_0)(t - t_0) + O((t-t_0)^2)$

$a'(t) = b(t)$

Lemma If f is C^1 on $\mathbb{R}$ $f : \mathbb{R} \to \mathbb{C}$

and $a : I \to M$ is C^1 / then

$t \mapsto \tau(f(a(t)))$ is C^1 on I

$$\frac{d}{dt} \tau(f(a(t))) = \tau(f'(a(t))\, a'(t))$$

Proof: Suppose $f(t) = t^n$

$$a(t+h)^n = \{a(t) + h\, b(t) + O(h^2)\}^n$$

$$= a(t)^n + h \sum_{k=0}^{n} a(t)^k\, b(t)\, a(t)^{n-k-1} + O(h^2)$$

$$\frac{a(t+h)^n - a(t)^n}{h} = \sum_{k=0}^{n} a(t)^k\, b(t)\, a(t)^{n-k-1} + O(h)$$

$$\tau\left(\frac{a(t+h)^n - a(t)^n}{h}\right) = \sum_{k=0}^{n} \tau(a(t)^k b(t) a(t)^{n-k-1})$$ 3

$$+ O(h)$$

$$= n\tau(a(t)^{n-1} b(t)) + O(h)$$

$$\frac{d}{dt}\tau(f(a(t))) = \tau(n a(t)^{n-1} b(t))$$

$$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!} \in M$$

$$= \tau(f'(a(t))\, a'(t))$$

Theorem (Fuglede & Kadison, 1951)

$\Delta(a) = e^{L(a)}$ $\quad \Delta(ab) = \Delta(a)\Delta(b)$

(i) $x \in M^{-1}$, $a \in M$

$$L(xe^a) = L(x) + \mathrm{Re}(\tau(a))$$

(ii) $\forall\, a, b \in M \quad L(ab) = L(a) + L(b)$

Proof (i) $\frac{d}{dt}e^{ta} = ae^{ta} = e^{ta}a$

Let $b(t) = \tau(\log(xe^{ta}e^{ta^*}x^*))$

Claim $\exists\ 0 < \alpha < \beta$ s.t.

$$\sigma(xe^{ta}e^{ta^*}x^*) \subseteq [\alpha, \beta] \text{ for } t \in [0,1]$$

Thus b is a C^1 function on $[0,1]$.

$$b'(t) = \tau\big((xe^{ta}e^{ta^*}x^*)^{-1}\{xe^{ta}ae^{ta^*}x^* + xe^{ta}a^*e^{ta^*}x^*\}\big)$$

$$= \tau(\cancel{x^{*-1}}\,\cancel{e^{-ta^*}}\,\cancel{e^{-ta}}\,\cancel{x^{-1}} \cdot \cancel{x}\,\cancel{e^{ta}}\,a\,\cancel{e^{ta^*}}\,\cancel{x^*})$$

$$+ \tau(\cancel{x^{*-1}}\,\cancel{e^{-ta^*}}\,\cancel{e^{-ta}}\,\cancel{x^{-1}} \cdot \cancel{x}\,\cancel{e^{ta}}\,a^*\,\cancel{e^{ta^*}}\,\cancel{x^*})$$

$$= 2\,\mathrm{Re}(\tau(a))$$

$$b(1) - b(0) = \int_0^1 b'(t)\,dt = 2\,\mathrm{Re}(\tau(a))$$

$$b(1) = b(0) + 2\,\mathrm{Re}(\tau(a))$$

$$\frac{1}{2}\left[\tau(\log(xe^a(xe^a)^*))\right] = \frac{1}{2}\left[\tau(\log(xx^*)) + 2\,\mathrm{Re}(\tau(a))\right]$$

$$\tau(\log|xe^a|) = \tau(\log|x|) + \mathrm{Re}(\tau(a))$$ 5

$$L(xe^a) = L(x) + \mathrm{Re}(\tau(a))$$

$$L(e^a) = L(1) + \mathrm{Re}(\tau(a)) = \mathrm{Re}(\tau(a))$$

$$L(e^{a_1}\cdots e^{a_n}) = \mathrm{Re}(\tau(a_1) + \cdots + \tau(a_n))$$

$$= L(e^{a_1}) + \cdots + L(e^{a_n})$$

Now assume $a, b \in M^{-1}$. Write

$a = u|a|$ $\quad$ u unitary

$u = e^{h_a}$ $\quad$ by Borel functional calc.

$h_a = \log(u)$

$|a| = e^{\log|a|}$ $\qquad a = e^{h_a} e^{\log|a|}$

$b = e^{h_b} e^{\log|b|}$

$$L(ab) = L(e^{h_a} e^{\log|a|} e^{h_b} e^{\log|b|})$$

$$= L(e^{h_a}) + L(e^{\log|a|}) + L(e^{h_b}) + L(e^{\log|b|})$$

$= L(a) + L(b).$

Suppose $a \in M$, $b \in M^{-1}$.

$a = u|a|$ (u unitary). Let

$a_\varepsilon = u\sqrt{a^*a+\varepsilon} \in M^{-1}$ for $\varepsilon > 0$

$\varepsilon \mapsto \sqrt{a^*a+\varepsilon}$ norm cont.

$\varepsilon \mapsto u\sqrt{a^*a+\varepsilon}\, b$ norm. cont

$\varepsilon \mapsto L(u\sqrt{a^*a+\varepsilon}\, b)$ u.s.c.

$$\overline{\lim_{\varepsilon \to 0^+}} L(u\sqrt{a^*a+\varepsilon}\, b) \leq L(u\sqrt{a^*a}\, b) = L(ab)$$

$$L(u\sqrt{a^*a+\varepsilon}\, b) = L(u) + L(\sqrt{a^*a+\varepsilon}) + L(b)$$
$$= 0 + \tfrac{1}{2} L(a^*a+\varepsilon) + L(b)$$
$$= L_\varepsilon(a) + L(b)$$

$$\therefore\ L(a) + L(b) = \lim_{\varepsilon \to 0} L(u\sqrt{a^*a+\varepsilon}\, b) \leq L(ab).$$

$0 = L(1) = L(bb^{-1}) = L(b) + L(b^{-1})$ 7

$\Rightarrow \quad L(b^{-1}) = -L(b)$

$$L(a) = L(abb^{-1}) \geq L(ab) + L(b^{-1})$$
$$= L(ab) - L(b)$$

$\Rightarrow \quad L(a) + L(b) \geq L(ab)$

Thus for $a \in M \quad b \in M^{-1}$

$$L(ab) = L(a) + L(b).$$

Suppose a, b arbitrary.

$a_\varepsilon = \sqrt{a^*a + \varepsilon} \in M^{-1}$, $a_\varepsilon b \to ab$ in norm & $L(a_\varepsilon b) = L(a_\varepsilon) + L(b)$

$$L(a) + L(b) = \inf_{\varepsilon > 0} \{ L_\varepsilon(a) + L(b) \}$$

$$= \inf_{\varepsilon > 0} L(a_\varepsilon b) \leq L(ab)$$

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$$L(a)+L(b) \le L(ab)$$

$$L(ab) = \frac{1}{2} L(abb^*a^*)$$

$$= \frac{1}{2} L(baa^*b^*)$$

$$\le \frac{1}{2} L(b(a^*a+\varepsilon)b^*)$$

$$= L(|a_\varepsilon| b) = L(a_\varepsilon) + L(b)$$

$$L(ab) \le \inf_{\varepsilon>0} L(a_\varepsilon) + L(b) = L(a) + L(b).$$

$$\Delta(a) = e^{L(a)} = \det(|a|)$$