1 April 2021 ◦ The Logarithm

- last week: $T \in M_N(\mathbb{C})$ with eigenvalues $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ repeated according to multiplicity,
- $\mu_T = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$, probability measure on $\mathbb{C}$ with support $= \sigma(T) = \{\lambda_i\}_{i=1}^{N}$.
- $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$\log(\det|x+iy-T|)$

$\swarrow$

Thm: $\int_{\mathbb{R}^2} \nabla^2 f(x,y) \operatorname{Tr}(\log|x+iy-T|)\, dx\, dy$

$$= 2\pi \sum_{i=1}^{N} f(\lambda_i)$$

for $f \in C_c^2(\mathbb{R})$ (compact support)

Thus $$\mu_T = \frac{1}{2\pi} \nabla^2 \operatorname{tr}(\log|\lambda - T|)$$

is the sense of distributional derivatives.

$\nabla^2 = \nabla \cdot \nabla \quad \overset{\det}{=} \Delta$

- (M, τ) finite von Neumann algebra

  $M =$ Banach space & $\mathbb{C}$ algebra with involution $\|a^* a\| = \|a\|^2$ $\quad \|ab\| \leq \|a\| \cdot \|b\|$

  $M \subseteq B(H)$ WOT closed

  $\tau : M \to \mathbb{C}$ WOT cont. $\tau(ab) = \tau(ba)$

  $\tau(1) = 1$, $\tau(a^* a) \geq 0$ with equality only if $a = 0$

  $(E(a) = \tau(a))$

Spectral Measures $\quad a \in M \subseteq B(H)$.

$a = a^*$ then $\exists\, \mu_a$ on $\sigma(a)$ $\subseteq [-\|a\|, \|a\|]$ such that for all bounded Borel functions

$$\tau(f(a)) = \int f(t)\, d\mu_a(t) \qquad X_*(P)$$

$$E(f(X)) = \int f(t)\, d\mu_X(t)$$

$f = \mathbb{1}_A \qquad$ LHS $= P(X \in A)$, RHS $\mu_X(A)$

If $a \in M$ $\quad |a| = \sqrt{a^* a} \geq 0.$ 3

$\mu_{|a|}$ = spectral measure of $|a|$

on $\sigma(|a|) \subseteq [0, \|a\|]$.

Let $L(a) = \int_{(1, \|a\|]} \log(t)\, d\mu_{|a|}(t)$

$+ \int_{[0,1]} \log(t)\, d\mu_{|a|}(t) = \int \log(t)\, d\mu_{|a|}(t)$

$= \tau(\log|a|)$

Brown Measure $\quad \mu_a = \frac{1}{2\pi} \nabla^2 \tau(\log|x+iy-a|)$

$= \frac{1}{2\pi} \nabla^2 \tau(\log|\lambda - a|)$

$\lambda = x+iy$

Main Thm

$$L(ab) = L(a) + L(b)$$

for all $a, b \in M$.

Lemma

(i) $\mu_{|a|} = \mu_{|a^*|}$

(ii) $L(a) = L(a^*)$

(iii) $L(a^*a) = 2L(a)$

(iv) if $0 \leq a \leq b$ then
$L(a) \leq L(b)$

(v) let $L_\varepsilon(a) = \frac{1}{2} L(a^*a + \varepsilon)$
then $L(a) = \inf_{\varepsilon > 0} L_\varepsilon(a)$ &

$L : M \to [-\infty, \infty)$ is
upper semi-continuous

Proof: $M \subseteq B(H)$. Given $a \in B(H)$
define $v(|a|\xi) = a\xi$ $\quad \xi \in H$

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$\| |a| \xi \|^2 = \| a \xi \|^2$. V is an isometry on $\overline{\operatorname{ran} |a|} = \ker(a)^\perp$

extend V to H by making V 0 on $\ker(a)$. $a = V|a|$

(polar decomposition). Note

$a^* = V^* \underline{(} V|a|V^*)$ $\quad V^*V = \operatorname{proj}_{\ker(a)^\perp}$

$p = \operatorname{proj}(\ker(a)^\perp)$ $\quad q = \operatorname{proj}(\overline{\operatorname{ran} a})$

$V^*V = p \quad VV^* = q \quad V \in M$

$\tau(1-p) = \tau(1 - V^*V) = \tau(1 - VV^*)$

$= \tau(1-q)$.

$\Rightarrow \exists\ w \in M$ s.t. $w^*w = 1-p$

$ww^* = 1-q$

Let $u = v + w$ is a unitary

$a = u|a| \qquad |a^*| = u|a|u^*$

$\tau(|a^*|^n) = \tau(|a|^n)$ ⑥

$\Rightarrow \mu_{|a|} = \mu_{|a^*|}$ This

proves (i),

(ii) $L(a^*) = \int \log(t)\, d\mu_{|a^*|}(t)$

$= \int \log(t)\, d\mu_{|a|}(t)$

(iii) Suppose $x \in M$, $x \geq 0$.

Let's show $\int f(t)\, d\mu_x(t)$

$= \int f(\sqrt{t})\, d\mu_{x^2}(t)$.

LHS $= \tau(f(x))$ $g(t) = \sqrt{t}$

RHS $= \tau(g(x^2))$

If $\log \notin L^1(\mu_{|a|})$ then

$L(a), L(a^*a) = -\infty$

If $\log \in L^1(\mu_{|a|})$ then ⑦

$$L(a) = \int \log(t)\, d\mu_{|a|}(t)$$
$$= \int \log(\sqrt{t})\, d\mu_{|a|^2}(t)$$
$$= \frac{1}{2} \int \log(t)\, d\mu_{a^*a}(t)$$
$$= \frac{1}{2} L(a^*a).$$

(iv) $0 \leq a \leq b \Rightarrow L(a) \leq L(b)$.

First a lemma in functional calculus.

Let $A = (t, \infty) \subseteq \mathbb{R}$. $\mathbb{1}_A(t) = 1$ if $t \in A$ & 0 otherwise

$e = \mathbb{1}_A(a)$

$f = \mathbb{1}_A(b)$. Lemma:

$\exists\, v \in M$ s.t. $v^*v = e$, $vv^* \leq f$.

$$e = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tau(e) = \frac{1}{3} \qquad \tau(f) = \frac{2}{3}$$

$$v = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v^* = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$v^*v = e \qquad vv^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leq f.$$

For $t > 0$ $\mu_a([0,t]) = 1 - \mu_a(A)$

$= 1 - \tau(e) \geq 1 - \tau(f) = 1 - \mu_b(A)$

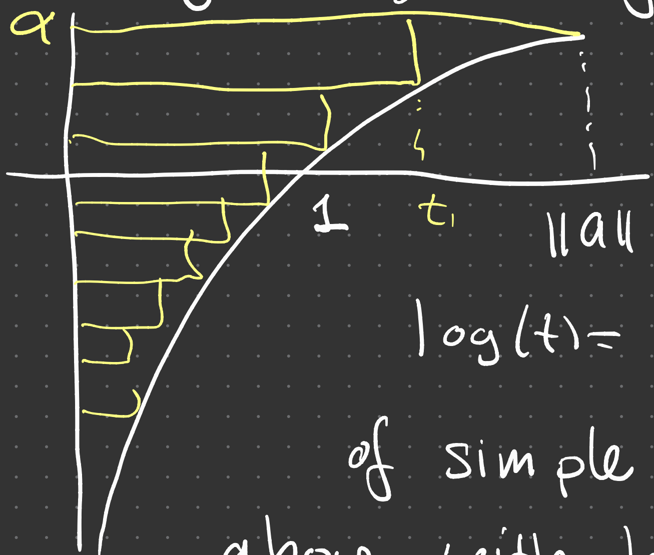
$= \mu_b([0,t])$.

$\mu_b([0,t]) \leq \mu_a([0,t])$

Let $s = \alpha + \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$ be a simple function of the form

$\alpha \in \mathbb{R}$ $\alpha_i \leq 0$, $A_i = [0, t_i]$

Then $\int s(t)\, d\mu_a(t) \leq \int s(t)\, d\mu_b(t)$



$\log(t) = \lim_{n\to\infty} S_n$ of simple fns as above with $\log(t) \leq S_{n+1}(t) \leq S_n(t)$

Thus $\int \log(t)\, d\mu_a(t)$

$= \lim_n \int S_n(t)\, d\mu_a(t) \leq \lim \int S_n(t)\, d\mu_b(t)$

$= \int \log(t)\, d\mu_a(t).$

Thus $L(a) \leq L(b)$.

(iv) Let $L_\varepsilon(a) = \frac{1}{2} L(a^*a+\varepsilon)$

$a \mapsto a^*a+\varepsilon : M \to M$
norm continuous. $M_+^{-1} \xrightarrow{\log} M$
is also norm continuous. So
$L_\varepsilon : M \to \mathbb{R}$ is norm cont.

As log is increasing

$$2L_\varepsilon(a) = \tau(\log(a^*a+\varepsilon))$$
$$= \int \log(t+\varepsilon)\, d\mu_{a^*a}(t)$$
$$\searrow \int \log(t)\, d\mu_{a^*a}(t)$$
$$= L(a^*a) = 2L(a).$$

Thus $L_\varepsilon(a) \searrow L(a)$ as $\varepsilon \to 0$.

So $\lim_{\varepsilon \to 0^+} L_\varepsilon(a) = L(a)$; also

for $0 < \varepsilon' < \varepsilon$ we have by (iv)

$L_{\varepsilon'}(a) \leq L_\varepsilon(a)$. Hence

$L(a) = \inf_{\varepsilon > 0} L_\varepsilon(a)$ and thus

L is upper semi-continuous.