Seminar on Brown Measure • 23 April 2021

- (M, τ) - finite von Neumann algebra, τ faithful normal trace, $\tau(1) = 1$.
- $a \in M$, $|a| = \sqrt{a^*a}$, $\mu_{|a|}$ = spectral measure of a,
  $\tau(f(|a|)) = \int_{[0,\infty)} f(t)\, d\mu_{|a|}(t)$
- $L(a) = \tau(\log(|a|)) = \int_{[0,\infty)} \log(t)\, d\mu_{|a|}(t)$
  $\in [-\infty, \infty)$
- $L_\varepsilon(a) = \frac{1}{2} L(a^*a + \varepsilon)$, $L(a) = \inf_{\varepsilon > 0} L_\varepsilon(a)$
- $L(ab) = L(a) + L(b) \quad \forall a, b \in M$

Goal for today: review of subharmonic functions. Goal for following lecture: show $\lambda \longmapsto L(a - \lambda 1)$ is subharmonic on $\mathbb{R}^2$.

- Harmonic fn's $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, f is harmonic $\nabla^2 f(x) = 0 \quad \forall x \in D$.

  if f is analytic on D, then $\mathrm{Re}(f)$ is harmonic e.g. $f(z) = \log|z|$ is harmonic on $\mathbb{C} \setminus \{0\}$.

Mean value property: if

$f: D \xrightarrow{C^2} \mathbb{R}$ harmonic $x \in D$ &

$B_r(x) \subseteq D$ then $f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x_1 + r\cos\theta, x_2 + r\sin\theta)\,d\theta$

$x = (x_1, x_2)$

$$M(f, x, r) = \frac{1}{2\pi}\int_0^{2\pi} f(x_1 + r\cos\theta, x_2 + r\sin\theta)\,d\theta$$

$$2\pi r \frac{d}{dr} M(f, x, r)$$

$$= r\int_0^{2\pi} \frac{\partial f}{\partial r}(x_1 + r\cos\theta, x_2 + r\sin\theta)\,d\theta$$

$$= \int_{\partial B_r(x)} \frac{\partial f}{\partial r}\,ds = \int_{\partial B_r(x)} \nabla f \cdot \vec{n}\,ds$$

$$= \int_{B_r(x)} \nabla^2 f(x)\, dx \quad \left(\begin{array}{c}\text{by} \\ \text{Green's Thm}\end{array}\right)$$

$$\boxed{2\pi r \frac{dM}{dr}(f, x, r) = \int_{B_r(x)} \nabla^2 f(x)\, dx}$$

If f is harmonic $\frac{dM}{dr}$ is 0.

Subharmonic Functions

$D \subseteq \mathbb{R}^2$ open $f : D \to [-\infty, \infty)$ is **subharmonic** if

(i) f is upper semi-continuous (usc)

(ii) $\forall x \in D$ $\forall r > 0$ s.t. $B_r(x) \subseteq D$
$f(x) \leq M(f, x, r)$

(iii) $\exists\ x_0 \in D$, s.t. $f(x_0) > -\infty$ ④

(on each conn. comp.)

Remark

Suppose f is sub-harmonic on D.

$x \in D$. $\forall \delta > 0$ $\exists \rho > 0$ s.t.

$B_\rho(x) \subseteq D$ & $f(y) \leq f(x) + \delta$

for $y \in B_\rho(x)$. Thus

$$2\pi M(f, x, r) = \int_0^{2\pi} f(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta$$

$$\leq 2\pi \left(f(x) + \delta\right) \quad (0 \leq r < \rho)$$

$\forall\ 0 \leq r < \rho$ $\quad M(f, x, r) \leq f(x) + \delta$.

$$\overline{\lim_{r \to 0^+}} M(f, x, r) \leq f(x).$$

$$f(x) \leq \underline{\lim_{r \to 0^+}} M(f, x, r)$$

$$\lim_{r \to 0^+} M(f, x, r) = f(x)$$

Suppose f is C^2 on $D \subseteq \mathbb{R}^2$

<u>Thm</u> f is subharmonic $\Leftrightarrow$ $\nabla^2 f(x) \geq 0$
$x \in D$

In fact

$$\lim_{r \to 0^+} \frac{M(f, x, r) - f(x)}{r^2} = \frac{1}{4} \nabla^2 f(x).$$

<u>Proof</u>: $$2\pi r \frac{dM}{dr} = \int_{B_r(x)} \nabla^2 f(y)\, dy$$

(i) Suppose $\nabla^2 f(x) \geq 0$. Then $\frac{dM}{dr} \geq 0$. Then $f(x) = \lim_{t \to 0^+} M(f, x, t)$
$\leq M(f, x, r) \Rightarrow f$ is subharmonic

(ii) $$M(f, x, r) - f(x) = \int_0^r \frac{d}{dt}\left[M(f, x, t)\right] dt$$

$$= \int_0^r \frac{1}{2\pi t} \int_{B_t(x)} \nabla^2 f(y)\, dy\, dt$$

$$= \frac{1}{4} \int_0^r 2t \left(\frac{1}{\pi t^2} \int_{B_t(x)} \nabla^2 f(y)\, dy \right] dt$$

$$= \frac{1}{4}\int_0^r 2t\, A(\nabla^2 f, x, t)\, dt$$ 6

$$g(x) = \lim_{r\to 0^+} \frac{1}{r^2}\int_0^r 2t\, A(g, x, t)\, dt$$

Thus $$\lim_{r\to 0^+} \frac{M(f,x,r) - f(x)}{r^2} = \frac{1}{4}\nabla^2 f(x).$$

So, if f is subharmonic $\nabla^2 f(x) \geq 0$.

The Laplacian of $\lambda \mapsto L(a-\lambda 1)$

Notation $\quad a_\lambda^* = a^* - \bar{\lambda}$

- $a_\lambda = a - \lambda \in M$, $\lambda \in \mathbb{R}^2$
- $\frac{\partial}{\partial\lambda} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial\bar{\lambda}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$
- $\nabla^2 = 4\frac{\partial^2}{\partial\bar{\lambda}\,\partial\lambda}$

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- $\frac{\partial}{\partial \lambda} a_\lambda = -1$, $\frac{\partial}{\partial \bar{\lambda}} a_\lambda = 0$
- $\frac{\partial}{\partial \lambda} a_\lambda^* = 0$, $\frac{\partial}{\partial \bar{\lambda}} a_\lambda^* = -1$

Lemma (i) $\frac{\partial}{\partial \lambda} \tau((a_\lambda^* a_\lambda)^n)$

$= -n\, \tau((a_\lambda^* a_\lambda)^{n-1} a_\lambda^*)$

(ii) $\frac{\partial}{\partial \bar{\lambda}} \tau((a_\lambda^* a_\lambda)^n a_\lambda^*)$

$= -\sum_{k=0}^{n} \tau((a_\lambda^* a_\lambda)^k (a_\lambda a_\lambda^*)^{n-k})$

Notation $L, R : M \to M$

$L(b) = a_\lambda^* a_\lambda b$, $R(b) = b a_\lambda a_\lambda^*$

$LR = RL$

Lemma Let $f_n(x) = x^n$. Then 8

$$\frac{\partial^2}{\partial\bar\lambda\partial\lambda} \tau(f_n(a_\lambda^* a_\lambda)) = -\tau\left(\frac{L f_n'(L) - R f_n'(R)}{L - R}(1)\right)$$

Proof:

$$\frac{\partial^2}{\partial\bar\lambda\partial\lambda} \tau((a_\lambda^* a_\lambda)^n) = -n \sum_{k=0}^{n-1} \tau\left((a_\lambda^* a_\lambda)^k 1 (a_\lambda a_\lambda^*)^{n-k-1}\right)$$

$$= -n \sum_{k=0}^{n-1} \tau\left(L^k R^{n-k-1}(1)\right)$$

$$= -n\,\tau\left(\frac{L^n - R^n}{L - R}(1)\right)$$

$$= -\tau\left(\frac{L f_n'(L) - R f_n'(R)}{L - R}(1)\right)$$

Remark

$0 \leq a$ & $\|a\| < K$ then $\|K-a\| \leq K$. Suppose $a \notin \mathbb{C}1$. Then $\|a_\lambda\| > 0 \ \forall \lambda$

Let $\delta_0 = \min_{|\lambda| \leq \frac{3\|a\|}{2}} \|a_\lambda\|$. If $|\lambda| > \frac{3\|a\|}{2}$

then $\|a_\lambda\| \geq \frac{1}{2}\|a\|$. $\|a_\lambda\|^2 \geq \delta > 0$ for all λ.

Now let $U \subseteq \mathbb{B}^2$ open

Choose $K > 0$ s.t.

$$0 < \delta \leq \|a_\lambda^* a_\lambda\| \leq K - \delta \text{ for } \lambda \in \overline{U}$$

$$\|K - a_\lambda^* a_\lambda\| < K - \delta \text{ for } \lambda \in \overline{U}$$

Let $f(x) = (x-K)^n$. Then 10

$$\left\| \frac{L f'(L) - R f'(R)}{L - R}(1) \right\|$$

$$\leq 2n(n+1) K (K-\delta)^{n-1}$$