Subharmonic Functions • Part II • 30 April 2021

- (M,τ) = finite von Neumann algebra, τ = trace
- $a \in M$, $L(a) = \tau(\log|a|) = \int \log(t)\, d\mu_{|a|}(t)$

  $|a| = \sqrt{a^*a}$, $L_\varepsilon(a) = \frac{1}{2}\tau(\log(a^*a+\varepsilon))$
- $L_\varepsilon(a)$ decreases to $L(a)$ as $\varepsilon \to 0^+$
- $L(ab) = L(a) + L(b)$ $\forall a, b \in M$.
- Goal: show $\lambda \mapsto L(a-\lambda)$ is subharmonic
- $f: D \to \mathbb{R}$, D open, f C^2, is harmonic if $\nabla^2 f(x) = 0$ for $x \in D$

$$M(f,x,r) = \frac{1}{2\pi}\int_0^{2\pi} f(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta$$

$$\nabla^2 f(x) = 4 \lim_{r\to 0^+} \frac{M(f,x,r) - f(x)}{r^2} \quad \left\{\text{for } C^2 \text{ functions}\right\}$$

- $f: D \to [-\infty, \infty)$ is sub-harmonic if
  - f is upper semi-continuous
  - $f(x) \le M(f,x,r)$ $\forall x \in D$
  - $\exists\, x_0$ with $f(x_0) > -\infty$

②

- if f is C^2 then f is sub-harmonic $\iff$ $\nabla^2 f(x) \geq 0 \ \forall x \in D$.

Example : $f(x) = \log\left(\sqrt{x_1^2 + x_2^2 + \varepsilon}\right)$

$\nabla^2 f(x) = 2\varepsilon\left(x_1^2 + x_2^2 + \varepsilon\right)^{-2} \geq 0$ for all $\varepsilon > 0$

$$\nabla^2 = 4\frac{\partial^2}{\partial\bar{\lambda}\,\partial\lambda} \qquad \frac{\partial}{\partial\lambda} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$$

$$\frac{\partial}{\partial\bar{\lambda}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$$

Show: $\dfrac{\partial^2}{\partial\bar{\lambda}\,\partial\lambda}\tau\left(\log\left(a_\lambda^* a_\lambda + \varepsilon\right)\right)$

$$= \varepsilon\,\tau\left(\left(a_\lambda^* a_\lambda + \varepsilon\right)^{-1}\left(a_\lambda a_\lambda^* + \varepsilon\right)^{-1}\right)$$

Recall $a_\lambda = a - \lambda 1 = a - \lambda$

$U \subseteq \mathbb{C}$ open , $\lambda \in U$.

Suppose : $\|a_\lambda^* a_\lambda\| < |\varepsilon|$ for $\lambda \in \overline{U}$.

$\varepsilon \in \{z \mid \operatorname{Re}(z) > 0\}$. $a_\lambda^* a_\lambda + \varepsilon$ invertible

$\log\left(a_\lambda^* a_\lambda + \varepsilon\right) = \log\left(\varepsilon\left(1 + \varepsilon^{-1} a_\lambda^* a_\lambda\right)\right)$

$$= \log(\varepsilon) + \log(1+\varepsilon^{-1}a_\lambda^* a_\lambda) \qquad \hat{3}$$

[because $\|\varepsilon^{-1} a_\lambda^* a_\lambda\| < 1$]

$$= \log(\varepsilon) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\varepsilon^{-1} a_\lambda^* a_\lambda)^n$$

$$\tau(\log(a_\lambda^* a_\lambda + \varepsilon)) = \log(\varepsilon) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \tau((\varepsilon^{-1} a_\lambda^* a_\lambda)^n)$$

$$= \log(\varepsilon) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\varepsilon^n} \tau((a_\lambda^* a_\lambda)^n)$$

uniformly on $\overline{U}$ & on cpt subsets of $B_{\|a_\lambda^* a_\lambda\|}(0)^c$

$$\frac{\partial^2}{\partial\bar\lambda\,\partial\lambda} \tau(\log(a_\lambda^* a_\lambda + \varepsilon))$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,\varepsilon^n} \frac{\partial^2}{\partial\bar\lambda\,\partial\lambda} \tau((a_\lambda^* a_\lambda)^n)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\cancel{n}\,\varepsilon^n}\, \cancel{n} \sum_{k=0}^{n-1} \tau((a_\lambda^* a_\lambda)^k (a_\lambda a_\lambda^*)^{n-k-1})$$

$$= \sum_{n=1}^{\infty} \sum_{\substack{k,l \geq 0 \\ k+l=n-1}} \varepsilon^{-1} (-\varepsilon^{-1})^k (-\varepsilon^{-1})^l \tau((a_\lambda^* a_\lambda)^k (a_\lambda a_\lambda^*)^l)$$

$$= \varepsilon^{-1} \sum_{n=1}^{\infty} \sum_{\substack{k,\ell \geq 0 \\ k+\ell = n-1}} \tau\left((-\varepsilon^{-1} a_\lambda^* a_\lambda)^k (-\varepsilon^{-1} a_\lambda a_\lambda^*)^\ell \right) \quad 4$$

$$= \varepsilon^{-1} \sum_{n=0}^{\infty} \sum_{\substack{k,\ell \geq 0 \\ k+\ell = n}} \tau\left((-\varepsilon^{-1} a_\lambda^* a_\lambda)^k (-\varepsilon^{-1} a_\lambda a_\lambda^*)^\ell \right)$$

$$= \varepsilon^{-1} \tau\left(\left[\sum_{k=0}^{\infty} (-\varepsilon^{-1} a_\lambda^* a_\lambda)^k \right] \left[\sum_{\ell=0}^{\infty} (-\varepsilon^{-1} a_\lambda a_\lambda^*)^\ell \right] \right)$$

$$= \varepsilon^{-1} \tau\left((1 - (-\varepsilon^{-1} a_\lambda^* a_\lambda))^{-1} (1 - (-\varepsilon^{-1} a_\lambda a_\lambda^*))^{-1} \right)$$

$$= \varepsilon \, \tau\left((a_\lambda^* a_\lambda + \varepsilon)^{-1} (a_\lambda a_\lambda^* + \varepsilon)^{-1} \right)$$

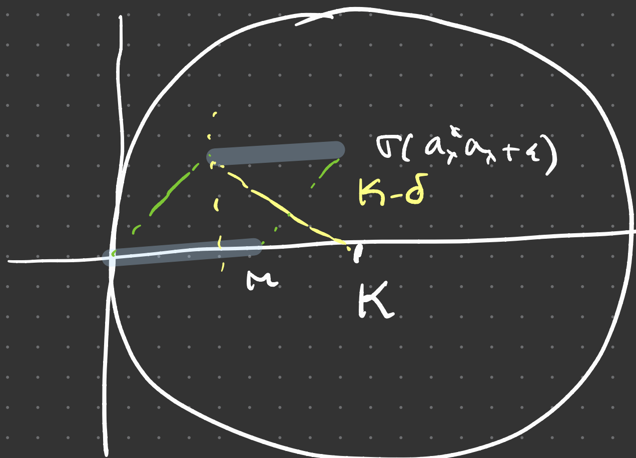
for $|\varepsilon| > \| a_\lambda^* a_\lambda \| \quad \lambda \in \overline{U}$

Show now that for all $\lambda \in \mathbb{C}$ and all ε s.t. $\mathrm{Re}(\varepsilon) > 0$

$$g(\lambda, \varepsilon) = \tau\left(\log(a_\lambda^* a_\lambda + \varepsilon) \right)$$

is C^∞ in λ and analytic in ε.

and $\frac{\partial^2 g}{\partial\bar{\lambda}\partial\lambda}(\lambda,\varepsilon)$ is analytic in ε.



$\lambda \in \overline{U}$

$\sigma(a_\lambda^* a_\lambda) \subseteq [0, M]$

$\exists\, \delta > 0$ s.t.

$\| a_\lambda^* a_\lambda + \varepsilon - K \| \leq K - \delta$

for $\lambda \in \overline{U}$, $\varepsilon \in \overline{V}$

$$\log(a_\lambda^* a_\lambda + \varepsilon) = \log(K) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{nK^n} (a_\lambda^* a_\lambda + \varepsilon - K)^n$$

converges uniformly & absolutely for $\lambda \in \overline{U}$ & $\varepsilon \in \overline{V}$

Convexity of Subharmonic functions ⑥

Suppose $D \subseteq \mathbb{C}$ open

$f : D \to [-\infty, \infty)$ subharmonic

$$M(f, x, r) = \frac{1}{2\pi}\int_0^{2\pi} f(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta$$

$0 < r_1 < r_2$ $\quad A_{r_1, r_2}(x) = \{ y \mid r_1 < \|x - y\| < r_2 \}$

$\subseteq D$. For $r_1 < r < r_2$

$r \mapsto M(f, x, r)$ is a convex function of $\log(r)$.

Proof: Suppose f is C^2. Then

$$0 \leq \int_{A_{r_1, r_2}(x)} \nabla^2 f(y)\, dy = \int_{\partial A} \nabla f \cdot \vec{n}\, ds$$

$$= \int_{\partial B_{r_2}(x)} \nabla f \cdot \vec{n}\, ds - \int_{\partial B_{r_1}(x)} \nabla f \cdot \vec{n}\, ds$$

$$= 2\pi \left[r_2 \frac{dM}{dr}(f, x, r_2) - r_1 \frac{dM}{dr}(f, x, r_1) \right]$$

Let $u(r) = 2\pi r \frac{dM}{dr}(f, x, r)$ ⑦

is an increasing function of r.

Let $v = \log(r)$ $\quad \frac{dM}{dv} = \frac{dM}{dr}\frac{dr}{dv} = r\frac{dM}{dr}$

$\Rightarrow$ M is a convex function of v.

Thus for $r < s < t$

$$\frac{M(f,x,s) - M(f,x,r)}{\log(s) - \log(r)} \leq \frac{M(f,x,t) - M(f,x,s)}{\log t - \log s}$$

For f C^2 we have:

$$2\pi r \frac{dM(f,x,r)}{dr} = \int_{B_r(x)} \nabla^2 f(y)\, dy.$$

So define $\mu_f(B_r(x)) = 2\pi r \frac{dM(f,x,r)}{dr}$

at all $x \in r$ at which

$\frac{dM(f,x,r)}{dr}$ exists. For each x this for almost all r.