Aug. 05. 2021

czwartek, 5 sierpnia 2021 15:23

Circular law for matrices with iid entries

$(X_{i,j})_{i,j\geq 0}$ $\qquad X_n = (X_{i,j})_{i,j=1}^{n}$

iid complex r.v.'s $\mathbb{E}X_{i,j} = 0 \quad Var(X_{i,j}) = 1$

$Var(X_{i,j}) = \mathbb{E}|X_{i,j}|^2 - |\mathbb{E}X_{i,j}|^2$

$$\mu_{X_n} \xrightarrow{w} \frac{1}{\pi} 1_{\{|z|\leq 1\}}$$

- $A \in M_n(\mathbb{C})$

  $\lambda_1(A), \dots, \lambda_n(A)$

  $|\lambda_1(A)| \geq \dots \geq |\lambda_n(A)|$ with growing phases.

- singular values are defined by

  $s_k(A) := \lambda_k(\sqrt{AA^*})$ for $1 \leq k \leq n$

  $s_1(A) \geq \dots \geq s_n(A) \geq 0$

Matrix $H_A = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ has eigenvalues

$$s_1(A), -s_1(A), \dots, s_n(A), -s_n(A)$$

$$\|A\|_{2\to 2} := \max_{\|x\|_2=1} \|Ax\|_2 = s_1(A)$$

$$s_n(A) = \min_{\|x\|_2=1} \|Ax\|_2$$

Thm 1.1 (Weyl inequalities)

For every $n\times n$ complex matrix A and $1\leq k\leq n$

$$\prod_{i=1}^{k} |\lambda_i(A)| \leq \prod_{i=1}^{k} s_i(A)$$

$$= \qquad \text{for } k = n$$

$$\to \prod_{i=1}^{n} |\lambda_i(A)| = |\det(A)| = \sqrt{\det(AA^*)} = \det(\sqrt{AA^*}) = \prod_{i=1}^{n} s_i(A)$$

we define empirical eigenvalues and singular values distributions

$$\mu_A = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k(A)} \qquad \nu_A = \frac{1}{n} \sum_{k=1}^{n} \delta_{s_k(A)}$$

$\mathbb{C}$ $\qquad$ $\mathbb{R}_+$

$$\int \log|\lambda| \, d\mu_A(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \log|\lambda_i(A)| = \frac{1}{n} \log|\det(A)| =$$

$$= \frac{1}{n} \sum_{i=1}^{n} \log(s_i(A)) = \int \log(s) \, d\nu_A(s)$$

$A \longmapsto (s_1(A), \dots, s_n(A))$ is 1-Lipschitz for operator norm and l^∞ norm

$$\max_{1 \le i \le n} |s_i(A) - s_i(B)| \le s_1(A-B) \quad \text{for } A, B \in M_n(\mathbb{C})$$

(Courant - Fischer)

Example

$$A_n = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \qquad B_n = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ K_n & & & 0 \end{pmatrix}$$

$(K_n)_n$ - sequence of positive numbers

$$\lambda_1(A_n) = \dots = \lambda_n(A_n) = 0$$

$$B_n^n = K_n I_n \qquad \lambda_k(B_n) = K_n^{1/n} e^{\frac{2k\pi i}{n}}$$

$B_n^n = K_n I_n \qquad \mu_K(B_n) = K_n \quad e$

if $K_n^{1/n} \to 1$ then $K_n = \left(1 - \frac{1}{\log(n)}\right)^n$

$\mu_{B_n} \xrightarrow{w}$ Uniform $\{z \in \mathbb{C} : |z| = 1\}$

$\mu_{A_n} \xrightarrow{w} \delta_0$

$AA^* = \operatorname{diag}(1, \dots, 1, 0)$ $\frac{1}{n}$

$BB^* = \operatorname{diag}(1, \dots, 1, K_n^2)$

hence

$\nu_{A_n} \xrightarrow{w} \delta_1$

$\nu_{B_n} \xrightarrow{w} \delta_1$

$X = (X_{ij})_{1 \le i,j \le n}$

eigenvalues of $\frac{1}{n} XX^* \longleftrightarrow$ singular values of $\frac{1}{\sqrt{n}} X$

Thm 2.1

Almost surely $\nu_{n^{-1/2} X} \xrightarrow{w} \mathcal{Q}$

$$\mathcal{Q}(dx) = \frac{\sqrt{4-x^2}}{\pi} \mathbb{1}_{[0,2]}(x)$$

quarter circular law.

Why $n^{-1/2}$ is correct normalization?

$$\int s^2 \, d\nu_{n^{-\frac{1}{2}} X}(s) = \frac{1}{n^2} \sum_{i=1}^{n} s_i^2(X) = \frac{1}{n^2} \operatorname{Tr}(XX^*) =$$

$$= \frac{1}{n^2} \sum_{i,j=1}^{n} |X_{i,j}|^2 \xrightarrow{a.s.} \mathbb{E}|X_{1,1}|^2$$

Thm 2.2

Thm 2.2

A.s. $\mu_{n^{-\frac{1}{2}}X} \xrightarrow{w} C_1$ as $n \to +\infty$

where C_1 is the uniform distr. on the unit disk.

Sketch of the proof of Thm 2.2.

(1) Using explicit formulas prove the theorem in the case $X_{1,1} = X + iY$ wher $(X,Y) \sim \mathcal{N}_2(0, \frac{1}{2} I_2)$

X, Y independent gaussian $\mathcal{N}(0, \frac{1}{2})$

$\mathcal{P}(\mathbb{C})$ - set of probability measures on $\mathbb{C}$, which integrate $\log|\cdot|$ in neighbourhood of infinity

The logarithmic potential of $\mu \in \mathcal{P}(\mathbb{C})$ is

$U_\mu : \mathbb{C} \to (-\infty, +\infty]$

$$U_\mu(z) = -\int_{\mathbb{C}} \log|z-\lambda| \, d\mu(\lambda)$$

$$U_{C_1}(z) = \begin{cases} -\log|z| & \text{if } |z| > 1 \\ \frac{1}{2}(1-|z|^2) & \text{if } |z| \leq 1 \end{cases}$$

Lemma 4.1

For every $\mu, \nu \in \mathcal{P}(\mathbb{C})$ if $U_\mu = U_\nu$ a.e. on $\mathbb{C}$ then $\mu = \nu$.

A - $n \times n$ complex matrix $P_A(z) = \det(A - zI)$ then

$$U_{\mu_A}(z) = -\int_{\mathbb{C}} \log|\lambda - z| \, d\mu_A(\lambda) = -\frac{1}{n} \log|\det(A - zI)|$$

$$= -\frac{1}{n} \log|P_A(z)|$$

$$= -\frac{1}{n}\log|P_A(z)|$$

for every $z \in \mathbb{C} \setminus \{\lambda_1(A), \dots, \lambda_n(A)\}$

$$U_{\mu_A}(z) = -\frac{1}{n}\log\det\left(\sqrt{(A-zI)(A-zI)^*}\right) = -\int_0^{+\infty}\log(t)\,d\nu_{A-zI}(t)$$

$$\boxed{\begin{array}{c}\text{Knowledge of } \nu_{A-zI} \text{ for a.a. } z \in \mathbb{C} \\ \Downarrow \\ \text{knowledge of } \mu_A\end{array}}$$

$(E, \mathcal{E})$

Borel function $f : E \to \mathbb{R}$ is uniformly integrable (u.i) for a seq. of prob. measures $(\eta_n)_{n\geq 1}$ on $\mathcal{E}$ when

$$\lim_{t\to\infty} \sup_{n\geq 1} \int_{\{|f|>t\}} |f|\,d\eta_n = 0$$

We use it:

if $\eta_n \xrightarrow{w} \eta$ as $n \to +\infty$

and if f is u.i. for $(\eta_n)_n$ then

f is η-integrable and $\lim_{n\to\infty}\int f\,d\eta_n = \int f\,d\eta$

Lemma 4.3 (Hermitization). *Let $(A_n)_{n\geq 1}$ be a sequence of complex random matrices where A_n is $n \times n$ for every $n \geq 1$. Suppose that there exists a family of (non-random) probability measures $(\nu_z)_{z\in\mathbb{C}}$ on $\mathbb{R}_+$ such that, for a.a. $z \in \mathbb{C}$, a.s.*

(i) $\nu_{A_n - zI} \rightsquigarrow \nu_z$ *as* $n \to \infty$

(ii) $\log$ *is uniformly integrable for* $(\nu_{A_n-zI})_{n\geq 1}$.

Then there exists a probability measure $\mu \in \mathcal{P}(\mathbb{C})$ such that

(j) *a.s.* $\mu_{A_n} \rightsquigarrow \mu$ *as* $n \to \infty$

(jj) *for a.a.* $z \in \mathbb{C}$,

$$U_\mu(z) = -\int_0^\infty \log(s)\,d\nu_z(s).$$

$M_n \qquad \nu_{M_n} \xrightarrow{w} \rho \qquad \left(\frac{X}{n^{1/2}} - zI\right)\left(\frac{X}{n^{1/2}} - zI\right)^*$

V_ϱ $\qquad V_{\frac{X}{n^{\frac{1}{2}}}} + M_n \longrightarrow V_\varrho$