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Circular law for matrices with iid entries

$A \in M_n(\mathbb{C})$

$\lambda_1, \dots, \lambda_n$ - eigenvalues $\mu_{A_n} = \frac{1}{n}(\delta_{\lambda_1} + \dots + \delta_{\lambda_n})$

$s_1 \geq \dots \geq s_n$ - singular values (eigenvalues of $\sqrt{AA^*}$) $\nu_{A_n} = \frac{1}{n}(\delta_{s_1} + \dots + \delta_{s_n})$

For Z - r.v. taking complex values $Var(Z) = \mathbb{E}(|Z|^2) - |\mathbb{E}Z|^2$

Our framework $(X_{ij})_{i,j \geq 1}$ infinite table of iid r.v.'s on $\mathbb{C}$ with $Var(X_{ij}) = 1$. $X = (X_{ij})_{i,j=1}^n$ $n^{-\frac{1}{2}}X$

$\nu_{n^{-\frac{1}{2}}X} \to Q_2$ - quarter circ.

$\mu_{n^{-\frac{1}{2}}X}$

$$\int s^2 \, d\nu_{n^{-\frac{1}{2}}X}(s) = \frac{1}{n^2} \sum_{i,j=1}^n |X_{ij}|^2 \xrightarrow{L.L.N.} \mathbb{E}(|X_{11}|^2)$$

Thm 2.2 Circular law

a.s. $\mu_{n^{-\frac{1}{2}}X} \xrightarrow{w} C_1$

$X_1, X_2, \dots$ $\quad \frac{\sum_{i=1}^n X_i}{n}$ $\quad X_1, \dots, X_1$ $\quad \frac{1}{n}\sum X_i = X_1$

Proof of Thm 2.2

1) Prove for X_{ij} complex Gaussian

$$U_\mu(z) = -\int_{\mathbb{C}} \log|z - \lambda| \, d\mu(\lambda)$$

$$U_{\mu_A}(z) = -\frac{1}{n} \log \det \sqrt{(A - zI)(A - zI)^*} = -\int_0^{+\infty} \log(t) \, d\nu_{A - zI}(t)$$

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f is uniformly integrable (U.I.) for a sequence $(\eta_n)_{n \geq 1}$ when

$$\lim_{t \to \infty} \sup_{n \geq 1} \int_{\{|f| > t\}} |f| \, d\eta_n = 0$$

$$\lim_{t \to \infty} \limsup_n \int_{\{|f| > t\}} |f| \, d\eta_n = 0$$

$$\lim_{t\to\infty} \limsup_{n} \int_{\{|f|>t\}} |f| \, d\eta_n = 0$$

$\eta_n \xrightarrow{w} \eta$ and f is U.I. for η_n then

f is η integrable and $\lim_{n\to\infty} \int f \, d\eta_n = \int f \, d\eta$

Lemma 4.3 (Hermitization). *Let $(A_n)_{n\geq 1}$ be a sequence of complex random matrices where A_n is $n \times n$ for every $n \geq 1$. Suppose that there exists a family of (non-random) probability measures $(\nu_z)_{z\in\mathbb{C}}$ on $\mathbb{R}_+$ such that, for a.a. $z \in \mathbb{C}$, a.s.*

(i) $\nu_{A_n - zI} \rightsquigarrow \nu_z$ *as* $n \to \infty$

(ii) $\log$ *is uniformly integrable for* $(\nu_{A_n-zI})_{n\geq 1}$.

$\xrightarrow{w}$

Then there exists a probability measure $\mu \in \mathcal{P}(\mathbb{C})$ such that

(j) *a.s.* $\mu_{A_n} \rightsquigarrow \mu$ *as* $n \to \infty$

(jj) *for a.a.* $z \in \mathbb{C}$,

$$\boxed{U_\mu(z)} = -\int_0^\infty \log(s) \, d\nu_z(s).$$

Moreover, if the convergence (i) and the uniform integrability (ii) both hold in probability *for a.a. $z \in \mathbb{C}$ (instead of for a.a. $z \in \mathbb{C}$, a.s.), then (j-jj) hold with the a.s. weak convergence in (j) replaced by the* weak convergence in probability.

Corollary 4.10

For all $z \in \mathbb{C}$ there exists a prob. measure ν_z depending only on z such that a.s.

$$\nu_{n^{-\frac{1}{2}}X - zI} \xrightarrow{w} \underline{\underline{\nu_z}} \quad \text{as } n \to +\infty$$

Lemma 4.11 (Count of small sing. values)

There exists $c_0 > 0$ and $0 < \gamma < 1$ such that a.s. for $n \gg 1$ and $\underline{\underline{n^{1-\gamma}}} \leq i \leq n-1$ and all $M \in M_n(\mathbb{C})$

$$\underline{\underline{s_{n-i}}}\left(n^{-\frac{1}{2}}X + M\right) \geq c_0 \frac{i}{n}$$

- There are at most $n^{1-\gamma}$ small singular values

Lemma 4.12

For every $a, d > 0$ there exists $b > 0$ such that if M is a deterministic complex $n \times n$ matrix with $\underline{s_1(M) \leq n^d}$ then

$$\mathbb{P}\left(s_n(X+M) \leq n^{-b}\right) \leq n^{-a}$$

In particular there exists $b > 0$ such that a.s. for $n \gg 1$

$$s_n(X+M) \geq n^{-b}$$

$$\int |\log(s)| \, d\nu_{n^{-\frac{1}{2}}X - zI}(s) \leq \int \frac{(\log(s))^2}{t} \, d\nu_{n^{-\frac{1}{2}}X - zI}(s) \leq \frac{1}{t} \int_0^{+\infty} s^p \nu_{n^{-\frac{1}{2}}X - zI}(s)$$

$$\int_{|\log(s)|>t} |\log(s)|\, d\nu_{n^{-1/2}X-zI}(s) \le \int \frac{(\log(s))^2}{t}\, d\nu_{n^{-1/2}X-zI}(s) < \frac{1}{t}\int_0^{+\infty} s^{p}\, \nu_{n^{-1/2}X-zI}(s) + \frac{1}{t}\int_0^{+\infty} s^{-p}\, \nu_{n^{-1/2}X-zI}(s)$$

$$\boxed{\frac{|\log(s)|}{t} > 1} \qquad (\log(s))^2 \le s^p + s^{-p} \quad p>0$$

So it suffices to show that

$$\limsup_{n\to\infty} \int s^{\pm p}\, d\nu_{n^{-1/2}X-zI}(s) < +\infty$$

Recall that $\max_{1\le i\le n} |s_i(A) - s_i(B)| \le s_1(A-B)$

$$A = (n^{-1/2}X - zI) \qquad B = n^{-1/2}X$$

$$|s_i(n^{-1/2}X - zI) - s_i(n^{-1/2}X)| \le |z|$$

$$s_i(n^{-1/2}X - zI) \le s_i(n^{-1/2}X) + |z|$$

So $\limsup \int s^p\, d\nu_{n^{-1/2}X-zI}(s) < +\infty$ for all $p \le 2$

$$\limsup_n \int s^{-p}\, d\nu_{n^{-1/2}X-zI}(s) < +\infty$$

$$\frac{1}{n}\sum_{i=1}^{n} s_i^{-p} \le \frac{1}{n}\sum_{i=1}^{n-\lfloor n^{1-\gamma}\rfloor} s_i^{-p} + \frac{1}{n}\sum_{i=n-\lfloor n^{1-\gamma}\rfloor+1}^{n} s_i^{-p}$$

$$\le \frac{1}{n} c_0^{-p} \sum_{i=1}^{n}\left(\frac{n}{i}\right)^p + \underbrace{\frac{1}{n} n^{1-\gamma}\cdot n^{pb'}}_{n^{pb'-\gamma}}$$

fine for $p<1$

$$s_n\left(\frac{1}{\sqrt{n}}X - zI\right) = \frac{1}{\sqrt{n}} s_n(X - \underbrace{z\sqrt{n}I}) \ge \frac{1}{\sqrt{n}} n^{-b} = n^{-\overbrace{(b+\frac{1}{2})}^{b'}}$$

$$pb' - \gamma < 0 \qquad p < \frac{\gamma}{b'}$$

for $p < \min\{1, \frac{\gamma}{b'}\}$ we get $\limsup \int s^{-p}\, d\nu_{n^{-1/2}X-zI}(s) < +\infty$

$$U_\mu(z) = -\int_0^{+\infty} \log(s)\, d\nu_z(s)$$

since ν_z does not depend on the distr. of $X_{i,j}$ it is as in the Gaussian case and

a.s. $U_{\mu} \xrightarrow{w} C$

as in the gaussian case and

a.s. $\mu_{n^{-\frac{1}{2}}X} \xrightarrow{w} C_1$

$\boxed{}$

Corollary 4.10

For all $z \in \mathbb{C}$ there exists a prob. measure ν_z depending only on z such that a.s.

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Pf.

Step 1

$$f : \mathbb{R} \to \mathbb{R}$$

$$\|f\|_{TV} = \sup \sum_{k \in \mathbb{Z}} |f(x_{k+1}) - f(x_k)|$$

if $f' \in L^1(\mathbb{R})$ $\quad \|f\|_{TV} = \int |f'(t)| \, dt$

Lemma 4.18 (Concentr. of measure for sing. val. emp. measure)
If M is an $n \times n$ complex random matrix with independent rows then for any $f : \mathbb{R} \to \mathbb{R}$ going to 0 at $\pm\infty$ with $\|f\|_{TV} \leq 1$ and every $t \geq 0$

$$\mathbb{P}\left(\left| \int f \, d\nu_M - \mathbb{E} \int f \, d\nu_M \right| \geq t \right) \leq 2 e^{-2nt^2}$$

Proof

Lidskii inequality: $A, B \in M_n(\mathbb{C})$ with $\operatorname{rank}(A-B) \leq k$ then $s_{i-k}(A) \geq s_i(B) \geq s_{i+k}(A)$

$F_A(\cdot) = \nu_A((-\infty, \cdot])$
$F_B(\cdot) = \nu_B((-\infty, \cdot])$
$$\|F_A - F_B\|_\infty \leq \frac{\operatorname{rank}(A-B)}{n}.$$