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Circular law for matrices with i.i.d. entries

Corollary 4.10

For all $z \in \mathbb{C}$ there exists a prob. measure ν_z depending only on z such that a.s.

$$\nu_{n^{-\frac{1}{2}}X - zI} \xrightarrow{w} \nu_z \quad \text{as } n \to +\infty$$

$$\mathbb{E}\, \nu_{n^{-\frac{1}{2}}X - zI}$$

Step 1

Lemma 4.18 (Concentr. of measure for sing. val. emp. measure)

If M is an $n \times n$ complex random matrix with independent rows then for any $f : \mathbb{R} \to \mathbb{R}$ going to 0 at $\pm\infty$ with $\|f\|_{TV} \le 1$ and every $t \geq 0$

$$\mathbb{P}\left(\left| \int f \, d\nu_M - \mathbb{E} \int f \, d\nu_M \right| \geq t \right) \le 2 e^{-2nt^2}$$

Proof

$$A, B \in M_n(\mathbb{C}) \quad \operatorname{rank}(A - B) \le k$$

$$s_{i-k}(A) \geq s_i(B) \geq s_{i+k}(A)$$

$$F_A(t) = \nu_A((-\infty, t])$$

$$F_B(t) = \nu_B((-\infty, t])$$

$$\| F_A - F_B \|_\infty \le \frac{\operatorname{rank}(A - B)}{n}$$

For a smooth $f : \mathbb{R} \to \mathbb{R}$

$$\left| \int f \, d\nu_A - \int f \, d\nu_B \right| = \left| \int f'(t) \left(F_A(t) - F_B(t) \right) dt \right| \le$$

$$\le \frac{\operatorname{rank}(A - B)}{n} \underbrace{\int |f'(t)| \, dt}_{= \|f\|_{TV}}$$

It follows that if f is measurable with $\|f\|_{TV} \le 1$ then

$$\left| \int f \, d\nu_A - \int f \, d\nu_B \right| \le \frac{\operatorname{rank}(A - B)}{n}$$

Fix f as above. For every $x_1, \dots, x_n$ in $\mathbb{C}^n$ denote by $A(x_1, \dots, x_n)$ the $n \times n$ matrix with rows $x_1, \dots, x_n$ define $F : (\mathbb{C}^n)^n \to \mathbb{R}$ by

$$F(x_1,\dots,x_n) = \int f \, d\nu_{A(x_1,\dots,x_n)}$$

For any $i \in \{1,\dots,n\}$ $x_i' \in \mathbb{C}^n$, we have

$$\operatorname{rank}\left(A(x_1,\dots,x_{i-1}, x_i, x_{i+1},\dots,x_n) - A(x_1,\dots,x_{i-1}, x_i', x_{i+1},\dots,x_n)\right) \le 1$$

thus

$$|F(x_1,\dots,x_{i-1}, x_i, x_{i+1},\dots,x_n) - F(\dots, x_i', \dots)| \le \frac{1}{n}$$

If $X_1,\dots,X_n$ are indep r.v.'s in $\mathcal{X}_1,\dots,\mathcal{X}_n$ and $f: \mathcal{X}_1 \times \dots \times \mathcal{X}_n \to \mathbb{R}$ is a measurable function then

$$\mathbb{P}\left(|f(X_1,\dots,X_n) - \mathbb{E} f(X_1,\dots,X_n)| \ge t \right) \le 2\exp\left(-\frac{2t^2}{c_1^2+\dots+c_n^2}\right)$$

for any $t \ge 0$

$$c_k = \sup_{x,x' \in D_k} |f(x) - f(x')| \quad \text{and} \quad D_k = \{(x,x') : x_i = x_i' \;\; i \ne k\}$$

$x \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n$

It is enough to look at $\mathbb{E}\, \nu_{n^{-\frac{1}{2}} X - z}$

Step 2

Hoffman-Wielandt ineq.

$$\sum_{n=1}^{n} (s_i(A) - s_i(B))^2 \le \|A-B\|_2^2 = \sum_{1 \le i,j \le n} |(A-B)_{i,j}|^2$$

Define

$$Y_{i,j} = X_{ij} \mathbb{1}_{\{|X_{i,j}| \le K\}} \qquad K = K_n, \quad K_n \to +\infty$$

$$Y = (Y_{ij})_{1 \le i,j \le n}$$

$$\frac{1}{n} \sum_{k=1}^{n} |s_k(n^{-\frac{1}{2}} Y - zI) - s_k(n^{-\frac{1}{2}} X - zI)|^2 \le$$

$$\le \frac{1}{n} \sum_{1 \le i,j \le n} \frac{1}{n} |X_{ij}|^2 I_{|X_{ij}| > K_n} = \frac{1}{n^2} \sum_{1 \le i,j \le n} |X_{ij}|^2 \mathbb{1}_{|X_{ij}| > K_n} \xrightarrow{\text{L.L.N.}} 0$$

By our assumptions $\mathbb{E} |X_{ij}|^2 I_{|X_{ij}| > K_n} \xrightarrow[n \to \infty]{} 0$

Wasserstein distance (W_2) between $\nu_{n^{-\frac{1}{2}} Y - zI}$ and $\nu_{n^{-\frac{1}{2}} X - zI}$

Convergence in W_2 implies weak convergence so it is enough to look at $\mathbb{E}\, \nu_{n^{-\frac{1}{2}}Y - zI}$

Centralization

$$Z_{ij} = Y_{ij} - \mathbb{E} Y_{ij} = Y_{i,j} - \mathbb{E} X_{11} \mathbb{1}_{\{|X_{1,1}| < K\}}$$

By Lidskii ineq

$$\max_{t \geq 0} \left| \nu_{n^{-\frac{1}{2}}Y - zI}([0,t]) - \nu_{n^{-\frac{1}{2}}Z - zI}([0,t]) \right| \leq \frac{\operatorname{rank}(Y - Z)}{n} \leq \frac{1}{n}$$

It is enough to prove that $\mathbb{E}\, \nu_{n^{-\frac{1}{2}}Z - zI} \xrightarrow{w} \nu_z$

In the following we allow the law of $X_{1,1}$ depend on n but we assume

$$\mathbb{E} X_{1,1} = 0 \qquad \mathbb{P}(|X_{11}| \geq K_n) = 0 \quad \text{and}$$

$$\mathbb{E} |X_{11}|^2 = \sigma_n^2 \qquad \text{and} \qquad K_n = o(\sqrt{n})$$

$$\sigma = \sigma_n \longrightarrow 1 \qquad \text{as } n \to +\infty$$

Step 3 Linearization

$$X \in M_n(\mathbb{C}) \qquad s_1(X), \dots, s_n(X)$$

$\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$ - eigenvalues are $\pm s_1(X), \dots, \pm s_n(X)$

By $\check{\nu}$ denote symmetrization of ν

$$\check{\nu}(A) = (\nu(A) + \nu(-A))/2$$

$$m_\nu(z) = \int \frac{1}{x - z}\, d\nu(x).$$

We have

$$\to m_{\check{\nu}_{n^{-\frac{1}{2}}X - zI}}(\eta) = \frac{1}{n} \operatorname{Tr}\left(H(z) - \eta I\right)^{-1} \leftarrow$$

where

$$H(z) = \begin{pmatrix} 0 & n^{-\frac{1}{2}}X - z \\ (n^{-\frac{1}{2}}X - z)^* & 0 \end{pmatrix}$$

Through a conjug with the perm. matrix $H(z)$ is eq. with the matrix

$$B(z) = B - q(z, 0) \otimes I_n$$

$$q(z, \eta) = \begin{pmatrix} \eta & z \\ \bar{z} & \eta \end{pmatrix}$$

$\eta \quad 1 \qquad 0 \quad X_{ij}$

$$B_{ij} = \frac{1}{\sqrt{n}} \begin{pmatrix} 0 & X_{ij} \\ \overline{X_{ji}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & n^{-\frac{1}{2}} X_{11} - z & n^{-\frac{1}{2}} X_{12} \\ 0 & 0 & n^{-\frac{1}{2}} X_{21} & n^{-\frac{1}{2}} X_{22} - z \\ n^{-\frac{1}{2}} \overline{X}_{11} - \overline{z} & n^{-\frac{1}{2}} \overline{X}_{21} & 0 & 0 \\ n^{-\frac{1}{2}} \overline{X}_{12} & n^{-\frac{1}{2}} \overline{X}_{22} - \overline{z} & 0 & 0 \end{pmatrix}$$

$$B(z) = \begin{pmatrix} 0 & n^{-\frac{1}{2}} X_{11} - z & 0 & n^{-\frac{1}{2}} X_{12} \\ n^{-\frac{1}{2}} \overline{X}_{11} - \overline{z} & 0 & n^{-\frac{1}{2}} \overline{X}_{21} & 0 \\ 0 & n^{-\frac{1}{2}} X_{21} & 0 & n^{-\frac{1}{2}} X_{22} - z \\ n^{-\frac{1}{2}} \overline{X}_{12} & 0 & n^{-\frac{1}{2}} \overline{X}_{22} - \overline{z} & 0 \end{pmatrix}$$

$$R(\eta) = (B(z) - \eta I_{2n})^{-1} = (B - q(z, \eta) \otimes I_n)^{-1}$$

$$\underline{m_{\tilde{\nu}_{n^{-1/2} X - zI}}(\eta)} = \frac{1}{2n} \operatorname{Tr}(R(\eta))$$

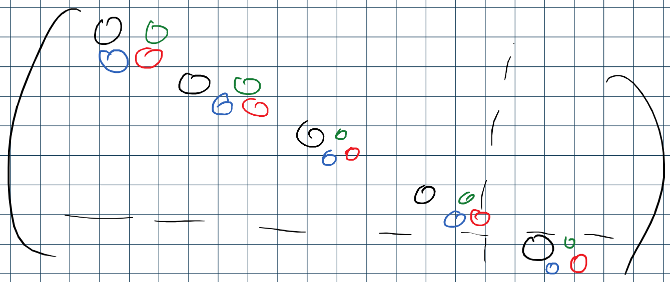
We set
$$R(\eta)_{kk} = \begin{pmatrix} a_k(\eta) & b_k(\eta) \\ c_k(\eta) & d_k(\eta) \end{pmatrix}$$

"It is easy to check that"

$$a(\eta) = \frac{1}{n} \sum_{k=1}^{n} a_k(\eta) \overset{!}{=} \frac{1}{n} \sum_{k=1}^{n} d_k(\eta)$$

$$b(\eta) = \frac{1}{n} \sum_{k=1}^{n} b_k(\eta) = \frac{1}{n} \sum_{k=1}^{n} \overline{c}_k(\eta)$$

$$R(\eta) = (B(z) - \eta I_{2n})^{-1} = -\frac{1}{\eta} \left(I - \frac{B(z)}{\eta} \right)^{-1} = -\frac{1}{\eta} \sum_{k=0}^{\infty} \left(\frac{B(z)}{\eta} \right)^k$$



Similar property holds for every power of $\underline{B(z)}$

$$m_{\tilde{\nu}_{n^{-1/2} X - zI}}(\eta) = a(\eta)$$

Want: $\mathbb{E}\, \underline{a(\eta)} \leftarrow$ converges