μ_x is symm.

Suppose that $x = x^*$. x is even

(assume that μ_x is NOT a Dirac Measure)

Let $\lambda > 0$. ($\lambda^2 \in \left(\frac{1}{\int \frac{1}{\omega^2} d\mu(\omega)}, \int \omega^2 d\mu(\omega) \right)$)

claim: $\exists\, v > 0$ such that $\lambda^2 v^2 = \dfrac{1 - \mathcal{M}_x(iv)}{\mathcal{M}_x(iv)}$

proof.

Let $f(v) = \mathcal{M}_x(iv) = \displaystyle\int_{\mathbb{R}} \frac{1}{1 - iv\omega} \frac{(1+iv\omega)}{(1+iv\omega)} d\mu_x(\omega)$

$$= \int_{\mathbb{R}} \frac{1}{1 + v^2\omega^2} d\mu_x(\omega) + \underbrace{i \int_{\mathbb{R}} \frac{v\omega}{1 + v^2\omega^2} d\mu_x(\omega)}_{=0}$$

$$= \int_{\mathbb{R}} \frac{1}{1 + v^2\omega^2} d\mu_x(\omega)$$

$\Rightarrow 0 < f < 1$

Define $g(v) = \dfrac{1 - f(v)}{v^2 f(v)}$ $\quad v \in (0, \infty)$

claim 1: g is strictly decreasing

Note that $g > 0$, so that $\log(g(v))$ is well-defined.

$$\frac{d}{dv}\log(g(v)) = \frac{g'(v)}{g(v)} = \frac{v^2 f(v)}{1-f(v)} \cdot \frac{-f'(v)(v^2 f(v)) - (1-f(v))\cdot(v^2 f)'}{(v^2 f(v))^2}$$

$$= -\frac{2(1-f(v))f(v) + v f'(v)}{v(1-f(v))f(v)}$$

It suffices to show that $2(1-f(v))f(v) + v f'(v) > 0$

$\left(\frac{d}{dv}\log(g(v)) < 0 \Rightarrow g' < 0 \Rightarrow g \searrow \right)$

$2(1-f(v))f(v) + v f'(v)$

$$= 2\cdot \int \frac{v^2 t^2}{1+v^2 t^2} d\mu(t) \cdot \int \frac{1}{1+v^2 s^2} d\mu(s)$$

$$+ v\cdot(-2v)\int \frac{t^2}{(1+v^2t^2)^2} d\mu(t)$$

$$\left(\int \frac{v^2 t^2}{1+v^2t^2} d\mu(t) + \int \frac{v^2 t^2}{1+v^2t^2} d\mu(t) \right) \int \frac{1}{1+v^2 s^2} d\mu(s)$$

$$= v^2 \cdot \int \frac{t^2+s^2}{(1+v^2t^2)(1+v^2s^2)} d\mu\times\mu(t,s)$$

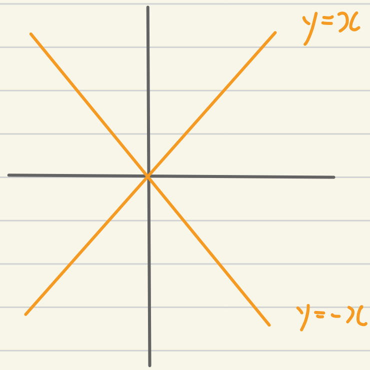
$$-v^2\cdot 2\int \frac{t^2}{(1+v^2t^2)^2} d\mu(t)$$

$$= -v^2 \cdot \int \left(\frac{t^2}{(1+v^2t^2)^2} + \frac{s^2}{(1+v^2s^2)^2} \right) d\mu\times\mu(t,s)$$

$$= v^4 \int \frac{(t^2-s^2)^2}{(1+v^2t^2)^2\,(1+v^2s^2)^2}\, d\mu\times\mu(t,s)$$

Since μ is NOT a Dirac Measure,

$\operatorname{supp}(\mu\times\mu) \not\subseteq \{(x,y) \mid |x|=|y|\}$



Thus, $2(1-f(v))f(v) + v f'(v) > 0.$

claim 2: $g((0,\infty)) = \left(\dfrac{1}{\int \frac{1}{w^2}\, d\mu(w)},\ \int w^2\, d\mu(w) \right)$

Note that

$$g(v) = \frac{1-f(v)}{v^2 f(v)} = \frac{\int \frac{v^2w^2}{1+v^2w^2}\, d\mu(w)}{\int \frac{v^2}{1+v^2w^2}\, d\mu(w)} = \frac{\int \frac{w^2}{1+v^2w^2}\, d\mu(w)}{\int \frac{1}{1+v^2w^2}\, d\mu(w)}$$

$\downarrow$ as $v \to \infty$

$$\frac{\int 1\, d\mu(w)}{\int \frac{1}{w^2}\, d\mu(w)}$$

$\downarrow$ as $v \to 0$

$$\frac{\int w^2\, d\mu(w)}{\int 1\, d\mu(w)}$$

$x = x^*$ even
$b = b^*$ Bernoulli $\Big\}$ x, b are free

$s = i\nu$ & $t = \frac{i}{\lambda^2 \nu}$

$x(s) = (1 - sx)^{-1} - \varphi\left((1-sx)^{-1}\right)$
$b(t) = (1 - t\lambda b)^{-1} - \varphi\left((1 - t\lambda b)^{-1}\right)$

$\rho = \left(M_x(s) \cdot M_{\lambda b}(t)\right)^{-1}$

known : $M_x(s) + M_{\lambda b}(t) = 1$

claim: $\mathcal{L}\left(1 - \rho\, x(s)\, b(t)\right) = 0.$

Fact
If a & b are free & $\varphi(a) = \varphi(b) = 0$
then $r(ab) = \varphi(a^*a)^{\frac{1}{2}} \varphi(b^*b)^{\frac{1}{2}}.$

proof.

check : $\varphi\left(x^*(s)\, x(s)\right) = \varphi\left(b^*(t)\, b(t)\right) = M_x(s) \cdot M_{\lambda b}(t) = \rho^{-1}$

$\Big($ $r\left(x(s)b(t)\right) = \varphi\left(x(s)^* x(s)\right)^{\frac{1}{2}} \varphi\left(b(t)^* b(t)\right)^{\frac{1}{2}} = \frac{1}{\rho}$
$\Rightarrow r\left(\rho\, x(s) b(t)\right) = \rho \frac{1}{\rho} = 1$ $\Big)$

$x^*(s) \cdot x(s) = \left((1 - \bar{s}x)^{-1} - \overline{\varphi\left((1 - sx)^{-1}\right)}\right) \cdot \left((1 - sx)^{-1} - \varphi\left((1 - sx)^{-1}\right)\right)$

$\Rightarrow \varphi\left(x^*(s)\, x(s)\right) = \varphi\left((1 + \nu^2 x^2)^{-1}\right) - |M_x(s)|^2$
$= M_x(s) - \left(M_x(s)\right)^2$
$= M_x(s)\left(1 - M_x(s)\right) = M_x(s)\, M_{\lambda b}(t) = \frac{1}{\rho}$

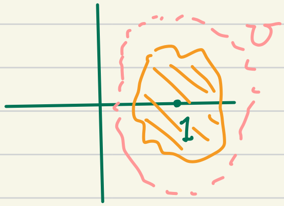
Lemma

If $\varphi(a^n) = 0$ for all $n \geq 1$ & $r(a) \leq 1$

then $\mathcal{L}(1-a) = 0$.

proof. Consider $r(a) < 1 \Rightarrow \sigma(1-a) \cap \{z \mid Re(z) \leq 0\} = \phi$

Hence log is analytic on open set U, $\sigma(1-a) \subseteq U$



$U = B_1(1)$

$$\left(\log(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n, \quad z \in U \right)$$

$$\Rightarrow \log(1-a) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-1)^n a^n$$

$$= -\sum_{n=1}^{\infty} \frac{a^n}{n}.$$

$$\varphi(\log(1-a)) = -\sum_{n=1}^{\infty} \frac{\overbrace{\varphi(a^n)}^{=0}}{n} = 0$$

$$\mathcal{L}(1-a) = \mathcal{L}\left(e^{\log(1-a)}\right) = Re\left(\varphi(\log(1-a))\right) = 0.$$

Now, We consider $r(a) = 1$, let $0 < t < 1$

$\Rightarrow \mathcal{L}(1-ta) = 0$

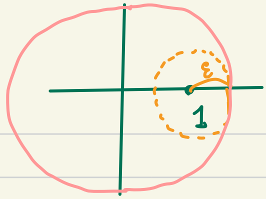
$\because z \mapsto \mathcal{L}(z-a)$ is subharmonic ($\Rightarrow$ u.s.c)

$\therefore \mathcal{L}(1-a) \geq \limsup_{t \to 1^-} \mathcal{L}(1-ta) = 0$

$\Rightarrow \mathcal{L}(1-a) \geq 0.$

$$k(z) \leq \frac{1}{2\pi} \int_0^{2\pi} k(z + \varepsilon e^{i\theta})\, d\theta$$

$k(z) = L(z-a)$



On the other hand, given $\varepsilon > 0$.

$$\mathcal{L}(1-a) \leq \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}(1+\varepsilon e^{j\theta}-a)\, d\theta$$

$$\leq$$

$$\mathcal{L}(1+\varepsilon e^{j\theta'}-a) \quad \text{for some } \theta' \in [0, 2\pi]$$

$$\leq$$

$$\max_{|z-1|\leq\varepsilon} \mathcal{L}(z-a) \leq \max_{|z|=1+\varepsilon} \mathcal{L}(z-a)$$

$$\mathcal{L}(z-a) = \mathcal{L}(z(1-z^{-1}a))$$

$$= \underbrace{\mathcal{L}(z)}_{=\log|z| = \log(1+\varepsilon)} + \underbrace{\mathcal{L}(1-z^{-1}a)}_{=0}$$

$$r(z^{-1}a) = \frac{1}{|z|} r(a) = \frac{1}{1+\varepsilon} r(a) < 1$$

$$\Longrightarrow \mathcal{L}(1-a) \leq \max_{|z|=1+\varepsilon} \mathcal{L}(z-a) = \log(1+\varepsilon) \to \log(1) = 0$$

as $\varepsilon \to 0^+$.