$b=b^*,\ b^2=1,\ \varphi(b)=0$

$x=x^*,\ x$ is even

Goal: if $a = bx$ is R-diagonal, then

$$\mu_a\left(B\left(0, \frac{1}{\sqrt{S_{a^*a}(t-1)}}\right)\right) = t \quad \forall t \in (\mu_x(\{0\}), 1]$$

Last Week

$a = bx$

$x^* = x$: even (all other moments vanish)

$b^* = b$: Bernoulli ($b^2=1$, $\varphi(b)=0$)

$$f(v) := M_x(iv) = \int \frac{1}{1+v^2w^2}\, d\mu_x(w) \qquad 0<f<1$$

generat. function for Boolean Cumulants (?)

$$g(v) := \frac{1-f(v)}{v^2 f(v)}, \quad v \in (0,\infty) \qquad g \searrow$$

$$\lambda := \lambda(v) = (g(v))^{\frac{1}{2}}, \qquad \lambda \in \Big(\underbrace{\Big[\int \frac{1}{w^2}\, d\mu(w)\Big]^{-1}}_{v\to\infty}, \underbrace{\int w^2\, d\mu(w)}_{v\to 0} \Big)$$

$$\mathcal{L}(a-\lambda) = \cancel{\mathcal{L}(b)}^{\,0} + \mathcal{L}(x+\lambda b)$$

$$= \int_{\mathbb{R}} \log(1+v^2w^2)\, d\mu(w) + \frac{1}{2}\log\left(\frac{\lambda^2}{1+v^2\lambda^2}\right)$$

Fact f, g, λ are analytic in a neighborhood of $(0,\infty)$

$\lambda \longmapsto \mathcal{L}(a-\lambda)$ is C^2 on $\mathbb{C}\setminus\{0\}$.

Notation

$$H(\lambda_1, \lambda_2) = \mathcal{L}(x + (\lambda_1 + i\lambda_2)b)$$

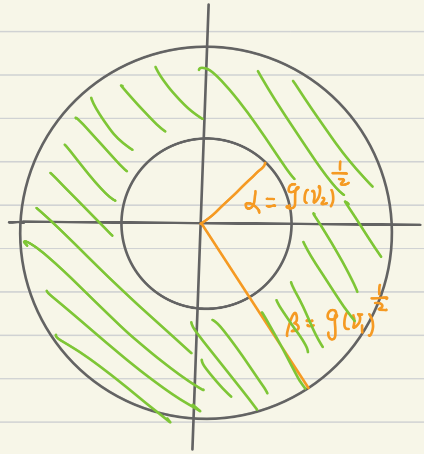
$$K(\lambda) = H(\lambda, 0) = \mathcal{L}(x + (\lambda + i0)b)$$

Note: $\mu_a = \frac{1}{2\pi} \nabla^2 \big(\underbrace{\log \Delta(a-\lambda)}_{= \mathcal{L}(a-\lambda)} \big)$ $\qquad \Delta(a) = e^{L(a)}$

Let $0 < \nu_1 < \nu_2 < \infty$ & then $\alpha = g(\nu_2)^{\frac{1}{2}}$. $\beta = g(\nu_1)^{\frac{1}{2}}$.

$$\mu_a \big([\alpha, \beta] \times_p [0, 2\pi) \big)$$

$$= \frac{1}{2\pi} \iint_{[\alpha,\beta] \times_p [0,2\pi)} \nabla^2 H(\lambda_1, \lambda_2) \, d\lambda_1 \, d\lambda_2$$



$$= \cdots = \beta \mathcal{K}'(\beta) - \alpha \mathcal{K}'(\alpha).$$

$$\mathcal{K}'(\lambda(\nu)) \lambda'(\nu) = \frac{d}{d\nu} \big[\mathcal{L}(x + \lambda(\nu) b) \big]$$

$$= \frac{d}{d\nu} \Big[\int_{\mathbb{R}} \log(1 + \nu^2 \omega^2) \, d\mu(\omega) + \frac{1}{2} \log \Big(\frac{\lambda^2}{1 + \nu^2 \lambda^2} \Big) \Big]$$

$$= \cdots = \frac{\lambda'(\nu) f(\nu)}{\lambda(\nu)}$$

$$\Rightarrow \mathcal{K}'(\lambda(\nu)) \lambda(\nu) = f(\nu)$$

$$\therefore \mu_a \big([\alpha, \beta] \times_p [0, 2\pi) \big) = f(\nu_1) - f(\nu_2)$$

If we let $\nu_1 \longrightarrow 0$ then $f(\nu_1) \longrightarrow 1$

$$\beta = g(\nu_1)^{\frac{1}{2}} \longrightarrow \int \omega^2 \, d\mu_x(\omega)$$

$\nu_2 \longrightarrow \infty$ then $f(\nu_2) \longrightarrow \mu_x(\{0\})$

$$\alpha = g(\nu_2)^{\frac{1}{2}} \longrightarrow \Big(\int \frac{1}{\omega^2} \, d\mu_x(\omega) \Big)^{-1}$$

Hence,

$$\mu_a\left(\left(\left(\int \frac{1}{\omega^2} d\mu_x(\omega)\right)^{-1}, \int \omega^2 d\mu_x(\omega)\right) \times_p [0, 2\pi)\right) = 1 - \mu_x(\{0\}).$$

<u>Case 1.</u> $\mu_x(\{0\}) = 0.$

$$\mathrm{supp}(\mu_a) = \left[\left(\int \frac{1}{\omega^2} d\mu_x(\omega)\right)^{-1}, \int \omega^2 d\mu_x(\omega)\right] \times_p [0, 2\pi)$$

<u>Case 2</u> $\mu_x(\{0\}) > 0.$

Let p be the orth. projection onto $\ker(a)$

$$\left(\ker(a) = \ker(bx) = \ker(x) \right)$$

$$\underset{\displaystyle 1_{\{0\}}(x)}{\underset{\shortparallel}{\varphi(p)}} = \int 1_{\{0\}}(\omega)\, d\mu_x(\omega) = \mu_x(\{0\}).$$

Note that $a = \begin{bmatrix} pap & pap^\perp \\ p^\perp a p & p^\perp a p^\perp \end{bmatrix} = \begin{bmatrix} 0 & R \\ 0 & S \end{bmatrix}$

$$\Rightarrow a - \lambda = \begin{bmatrix} -\lambda & R \\ 0 & S-\lambda \end{bmatrix}$$

Remark

$$\Delta\left(\begin{bmatrix} A_1 & B \\ & A_2 \end{bmatrix}\right) = \Delta\left(\begin{bmatrix} 1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \Delta\left(\begin{bmatrix} 1 & 0 \\ 0 & A_2 \end{bmatrix}\right) \Delta\left(\begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}\right) \Delta\left(\begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

Subclaim: $\Delta\left(\begin{bmatrix} 1 & B \\ & 1 \end{bmatrix}\right) = 1$

$$\ulcorner \quad e^{\begin{bmatrix} 0 & B \\ & 0 \end{bmatrix}} := \sum_{n=0}^{\infty} \frac{1}{n!}\left(\begin{bmatrix} 0 & B \\ & 0 \end{bmatrix}\right)^n = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \frac{1}{1!}\begin{bmatrix} 0 & B \\ & 0 \end{bmatrix} = \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \Delta\left(\begin{bmatrix} 1 & B \\ & 1 \end{bmatrix}\right) = \Delta\left(e^{\begin{bmatrix} 0 & B \\ & 0 \end{bmatrix}}\right) = \exp\left(\mathrm{Re}\left\{\varphi\left(\begin{bmatrix} 0 & B \\ & 0 \end{bmatrix}\right)\right\}\right) = 1$$

$$\mathcal{L}\left(\begin{bmatrix} 1 & 0 \\ 0 & A_2 \end{bmatrix}\right) = \varphi\left(\log\left|\begin{bmatrix} 1 & 0 \\ 0 & A_2 \end{bmatrix}\right|\right)$$

$$= \varphi\left(\log\left(\begin{bmatrix} 1 & 0 \\ 0 & |A_2| \end{bmatrix}\right)\right)$$

$$= \int \log(t)\, d\mu_{\left|\begin{bmatrix} 1 & 0 \\ 0 & |A_2| \end{bmatrix}\right|}(t)$$

$$= \int \log(t)\left(\delta_1(t)\,\varphi(p) + \mu_{|A_2|}(t)\cdot\varphi(1-p)\right)$$

$$= \varphi(1-p)\cdot\int \log(t)\, d\mu_{|A_2|}(t) = \varphi(1-p)\,\mathcal{L}(A_2)$$

$$\mathcal{L}\left(\begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \varphi(p)\,\mathcal{L}(A_1)$$

Hence, $\mathcal{L}(a-\lambda) = \varphi(p)\,\mathcal{L}_1(-\lambda) + \varphi(1-p)\,\mathcal{L}_2(s-\lambda)$

$$\Rightarrow \mu_a = \varphi(p)\cdot\delta_0 + \varphi(1-p)\,\mu_s$$

Observe that

$$\mu_a(\{0\}) \geq \varphi(p) = \mu_x(\{0\})$$

$$1 \geq \mu_a\left([0, \int \omega^2 d\mu_x(\omega)) \times_p [0, 2\pi) \right)$$

$$\geq \mu_a(\{0\}) + \mu_a\left(\left(\left[\int \tfrac{1}{\omega^2} d\mu_x(\omega)\right]^{-1}, \int \omega^2 d\mu_x(\omega) \right) \times_p [0, 2\pi) \right)$$

$$\geq \mu_a(\{0\}) + (1 - \mu_x(\{0\})$$

$$= 1 + (\mu_a(\{0\}) - \mu_x(\{0\})) \geq 1$$

$$\Rightarrow \mu_a(\{0\}) = \mu_x(\{0\}).$$

Furthermore, $\left(\int \frac{1}{\omega^2} d\mu_x(\omega) \right)^{-1} = 0$ &

$$\operatorname{supp}(\mu_a) = [0, \int \omega^2 d\mu_x(\omega)] \times_p [0, 2\pi).$$

Now, for $v > 0$.

$$\mu_a(B(0, \lambda(v))) = 1 - \mu_a\left([\lambda(v), \int \omega^2 d\mu_x(\omega)) \times_p [0, 2\pi) \right)$$

$$= 1 - (1 - f(v)) = f(v).$$

$$f(v) - 1 = \int \frac{-v^2\omega^2}{1 + v^2\omega^2} d\mu_x(\omega) = \int \frac{-v^2\omega}{1 - (-v^2)\omega} d\mu_{x^2}(\omega)$$

$$= \tilde{\mu}_{x^2}(-v^2) = \tilde{\mu}_{a^*a}(-v^2)$$

provide that v small

$$\Longrightarrow \quad f(\nu)-1 = \tilde{M}_{a^*a}(-\nu^2) \quad \Longrightarrow \quad \tilde{M}_{a^*a}^{\langle -1\rangle}(f(\nu)-1) = -\nu^2.$$

Then
$$S_{a^*a}(f(\nu)-1) = \frac{(f(\nu)-1)+1}{f(\nu)-1}\,\tilde{M}_{a^*a}^{\langle -1\rangle}(f(\nu)-1)$$

$$= \frac{f(\nu)}{f(\nu)-1}(-\nu^2) = \frac{1}{\lambda(\nu)^2}.$$

Hence,
$$\mu_a\left(B\left(0, S_{a^*a}(f(\nu)-1)^{-\frac{1}{2}}\right)\right) = \mu_a\left(B(0, \lambda(\nu))\right)$$

$$= f(\nu).$$

$$\xRightarrow[f(\nu)=t]{} \quad \mu_a\left(B\left(0, S_{a^*a}(t-1)^{-\frac{1}{2}}\right)\right) = t.$$