# Representations of R-diagonal Operators

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$(A, \varphi)$ = non-commutative probability space $\left[\begin{array}{l}\varphi \text{ is a trace} \\ \varphi(ab) = \varphi(ba)\end{array}\right]$

$$K_1(a) = \varphi(a)$$

$$K_2(a, a) = \varphi(a^2) - \varphi(a)^2$$

$$K_3(a, a, a) = \varphi(a^3) - 3\varphi(a)\varphi(a^2) + 2\varphi(a)^3$$

**Product Rule:** $r \geq 2$, $n_1 + \cdots + n_r = n$

$$K_r(a_1 \cdots a_{n_1}, a_{n_1+1} \cdots a_{n_1+n_2}, \ldots, a_{n_1+\cdots+n_{r-1}+1} \cdots a_{n_1+\cdots n_r})$$

$$= \sum_{\substack{\pi \in NC(n) \\ \pi \vee \rho = 1_n}} K_\pi(a_1, \ldots, a_n)$$

$$\rho = \{(1, \ldots, n_1), (n_1+1, \ldots, n_1+n_2), \ldots, (n_1+\cdots+n_{r-1}+1, \ldots, n_1+\cdots+n_r)\}$$

$$\underbrace{\sqcup\!\sqcup\!\sqcup\!\sqcup}_{n_1} \quad \underbrace{\sqcup\!\sqcup\cdots\sqcup}_{n_2} \quad \cdots \quad \underbrace{\sqcup\!\sqcup\!\sqcup\cdots\sqcup}_{n_r} \in NC(n)$$

$p = \sqcup \; | \; | \cdots |$

$\pi =$ 

## R-Diagonal Operator

$a \in (\mathcal{A}, \varphi)$ a $*$-probability sp

$\varphi(a^* a) \geq 0$

$\varphi(a^*) = \overline{\varphi(a)}$

is R-diagonal if

$K_n(a^{(\varepsilon_1)}, \ldots, a^{(\varepsilon_n)}) = 0$ unless

$n$ is even & $\varepsilon_i = -\varepsilon_{i+1}$ $\quad 1 \leq i \leq n-1$

$a^{(-1)} = a^*, \; a^{(1)} = a \quad \varepsilon_1, \ldots \varepsilon_n \in \{-1, 1\}$

$K_1(a) = 0, \quad K_2(a, a) = 0,$

$K_3(a, a^*, a) = 0, \ldots$

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$c = \frac{1}{\sqrt{2}}(x_1 + i x_2)$ circular

$x_1, x_2$ free & semi-circular

$\kappa_2(c, c^*) = \kappa_2(c^*, c) = 1$ all other $*$-cumulants are $0$.

$u$ = Haar unitary, $\varphi(u^n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$

$u^* = u^{-1}$

$\kappa_n(u, u^*, \dots, u, u^*) = (-1)^{n/2 - 1} C_{n/2 - 1}$

Theorem Let $a$ be R-diagonal and $\alpha_n = \kappa_{2n}(a, a^*, \dots, a, a^*)$
$= \kappa_{2n}(a^*, a, \dots, a^*, a)$. Then

$$\kappa_n(a^*a, \dots, a^*a) = \sum_{\sigma \in NC(n)} \alpha_\sigma .$$

**Proof:** Let $\rho = \{(1,2), \ldots, (2n-1, 2n)\}$

$=$  Then

$$\kappa_n(a^*a, \ldots, a^*a) = \sum_{\substack{\pi \in NC(2n) \\ \pi \vee \rho = 1_{2n}}} \kappa_\pi(a^*, a, \ldots, a^*, a)$$

$$NC(n) \ni \sigma \longmapsto \pi \in NC(2n)$$

Let $V \in \sigma$ $\qquad \pi \vee \rho = 1_{2n}$

$V = (i_1, \ldots, i_k)$ $\qquad$ & $\kappa_\pi(a^*, a, \ldots, a^*, a) = \alpha_\sigma$

$V \mapsto W = (2i_1, 2i_1+1, \ldots, 2i_k, 2i_k+1)$

$\pi$ is the partition with blocks $W$ obtained from $\sigma$ as above.

Suppose $2k+1 \in W \in \pi$. Claim: $2k \in W$. For simplicity suppose $2k+1 = 1$. Let $\ell$ be the last element of $W$.

$$|W| + |\text{sum of gaps}| = \ell$$


sum of $|\text{gaps}|$ is even $\Rightarrow$ $l$ is even. If $l < 2n$, then the blocks

$(l+1, l+2) \dots (2n-1, 2n)$ are not connected by $\pi$ to 


We must have $2n \in \overline{W}$. Thus we have for every $k$, $2k+1$ & $2k$ are in the same block of $\pi$.

Thus $\exists\, \sigma \in NC(n)$ st $\sigma \mapsto \pi$ under the mapping and

$$\alpha_\sigma = \kappa_\pi(a^*, a, \dots a^* a).$$

**Theorem** Suppose $b$ & $x$ are free, $b$ is Bernoulli: $b = b^*$, $b^2 = 1$, $\varphi(b) = 0$, $x = x^*$ is *even* $\varphi(x^{2n-1}) = 0$ $\forall n \geq 1$. Then $a = bx$ is R-diagonal.

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**Proof:** Let $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$

Suppose $\kappa_n(a^{(\varepsilon_1)}, a^{(\varepsilon_2)}, \dots, a^{(\varepsilon_n)}) \neq 0$.

We'll show:
- $n$ is even
- $\varepsilon_i = -\varepsilon_{i+1}$, $1 \leq i \leq n-1$.

Let us first show that $n$ is even.

$$\kappa_n(a^{(\varepsilon_1)}, \dots, a^{(\varepsilon_n)})$$

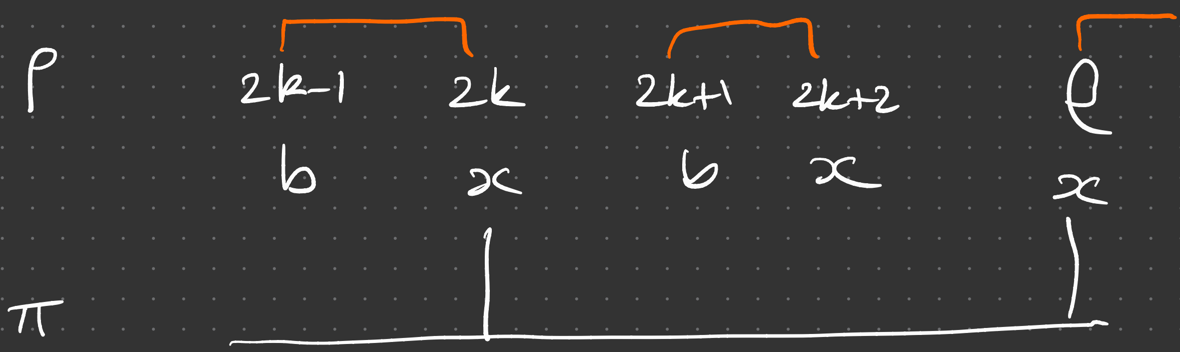
$$= \sum_{\substack{\pi \in NC(2n) \\ \pi \vee \rho = 1_{2n}}} \kappa_\pi(b, x, \dots, x, b, \dots, b, x, \dots)$$

$\rho = \sqcup \sqcup \sqcup \cdots \sqcup$

Suppose $\kappa_\pi(b, x, \ldots) \neq 0$.

Since $b$ & $x$ are free and mixed cumulants vanish $\pi$ must consist of "$x$-blocks" and "$b$-blocks".

So n° $x$'s is $n$ and is also the sum of lengths of "$x$-blocks" but each $x$-block is even. Thus $n$ is even. Suppose $\exists k$ s.t. $\varepsilon_k = \varepsilon_{k+1} = 1$




Let $V$ be the block containing $2k$ and $\ell$ be the next element.

The gap between $2k$ & $l$ (8) must consist of $x$-blocks & $b$-blocks both of which are even thus $l$ is odd. Thus either $l = 2k+1$ or $2k$ is the last element of $V$. We cannot have $l = 2k+1$. So $2k$ must be the last element. So we let $l$ be the first element. This will also create a disconnection so we must have $\varepsilon_k = -\varepsilon_{k+1}$. Thus $a$ is $R$-diagonal.

# Remark



Suppose $x \geq 0$

$$y = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \quad |y| = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

$y$ is an even op. with $|y|$ having the same dist as $x$.

If $b$ & $y$ are free with $b$ Bernoulli then $a = by$ is R-diagonal

$(by)^*(by) = yb^2y = y^2$ has dist as $x^2$.