The S-transform. 18 June 2021

$a \in (\mathcal{A}, \varphi) \quad m_n = \varphi(a^n)$

$\kappa_n = \kappa_n(a, \ldots, a)$

$(*) \quad m_n = \sum_{\pi \in NC(n)} \kappa_\pi \,, \quad \kappa_n = \sum_{\pi \in NC(n)} \mu(\pi, 1_n)\, m_\pi$

$\mu(\pi, 1)$ is a product of signed Catalan numbers $\beta_n = (-1)^{n-1} C_{n-1}$

$\mu($  $, 1_n)$

$= \mu(0_n,$  $) = \beta_1^2 \beta_2^3$

$$M(z) = 1 + \sum_{n=1}^{\infty} m_n z^n$$

$$C(z) = 1 + \sum_{n=1}^{\infty} \kappa_n z^n$$

$$(*) \Rightarrow m_n = \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \geq 0 \\ i_1+\dots+i_s = n-s}} \kappa_n m_{i_1} \cdots m_{i_s}$$

$$\Rightarrow \quad M(z) = C(zM(z)) \qquad (**)$$

$$\tilde{M}(z) = M(z) - 1, \quad \tilde{R}(z) = C(z) - 1$$

$$f(z) = zM(z)$$

$$(**) \Rightarrow \quad \tilde{M} = \tilde{R} \circ f$$

Inversion of Formal Power Series

$$f(z) = e^z - 1 = \sum_{n \geq 1} \alpha_n z^n \qquad \alpha_n = \frac{1}{n!}$$

$$g(z) = \log(1+z) = \sum_{n \geq 1} \beta_n z^n, \quad \beta_n = \frac{(-1)^{n-1}}{n}$$

$$f \circ g(z) = g \circ f(z) = z$$

In general $f(z) = \alpha_1 z + \alpha_2 z^2 + \dots$

$$f(0) = 0, \quad f'(0) = \alpha_1 \neq 0$$

Then we can find $g(z)$ ③

$= \beta_1 z + \beta_2 z^2 + \cdots$

such that $g(0) = 0$, $g'(0) = \beta_1 = \frac{1}{\alpha_1}$

& $f(g(z)) = g(f(z)) = z$

$$z = f(g(z))$$

$$= \alpha_1(\beta_1 z + \beta_2 z^2 + \cdots) + \alpha_2(\beta_1 z + \beta_2 z^2 + \cdots)^2 + \cdots$$

$$\alpha_1 \beta_1 = 1$$

$$\alpha_1 \beta_2 + \alpha_2 \beta_1^2 = 0$$

$$\alpha_1 \beta_3 + 2\alpha_2 \beta_1 \beta_2 + \alpha_3 \beta_1^3 = 0$$

Suppose $m_1 \neq 0$, then $k_1 = m_1 \neq 0$

so $\tilde{R}$ & $\tilde{M}$ have inverses.

$\tilde{R}(z) = k_1 z + k_2 z^2 + \cdots$

$\tilde{M}(z) = m_1 z + m_2 z^2 + \cdots$

$$\tilde{M} = \tilde{R} \circ f \Rightarrow \tilde{R}^{\langle -1 \rangle} = f \circ \tilde{M}^{\langle -1 \rangle}$$

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Let $\tilde{S} = \tilde{R}^{\langle -1 \rangle}$ $\quad S(z) = \dfrac{\tilde{S}(z)}{z}$

$$= \frac{1}{K_1} - \frac{K_2 z}{K_1^3} + \frac{2K_2^2 - K_1 K_2}{K_1^4} z^2 + \cdots$$

$$f(z) = z(1 + \tilde{M}(z))$$

$$f(\tilde{M}^{\langle -1 \rangle}(z)) = \tilde{M}^{\langle -1 \rangle}(z)\left(1 + \tilde{M}(\tilde{M}^{\langle -1 \rangle}(z))\right)$$

$$= (1+z)\, \tilde{M}^{\langle -1 \rangle}(z)$$

$$\tilde{S}(z) = (1+z)\, \tilde{M}^{\langle -1 \rangle}(z)$$

$$\boxed{S_a(z) = \frac{1+z}{z} \tilde{M}_a^{\langle -1 \rangle}(z)}$$

Definition S_a is the S-transform of a.

Goal: If a & b are free & $\varphi(a) \neq 0$, $\varphi(b) \neq 0$ then

$$S_{ab} = S_a S_b .$$ ⑤

Examples

$a = 1$

$$\tilde{M}_1(z) = \varphi(1)z + \varphi(1^2)z^2 + \cdots = \frac{z}{1-z}$$

$$\tilde{M}_1^{\langle -1 \rangle}(z) = \frac{z}{1+z} \qquad S_1(z) = \frac{1+z}{z}\tilde{M}_1^{\langle -1\rangle}(z) = 1$$

$a = p$ $\quad p = p^*,\ p^2 = p$. Let

$$\alpha = \varphi(p), \quad 0 < \alpha < 1 .$$

$$\tilde{M}_p(z) = \alpha z + \alpha z^2 + \cdots = \frac{\alpha z}{1-z}$$

$$\tilde{M}_p^{\langle -1 \rangle}(z) = \frac{z}{\alpha + z} . \quad S_p(z) = \frac{1+z}{z} \cdot \frac{z}{\alpha + z}$$

$$= \frac{1+z}{\alpha + z}$$

a^*a, with a circular

$$K_n(a^*a, \dots, a^*a) = 1 \quad \forall n \geq 1$$

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$$\tilde{R}(z) = z + z^2 + \cdots = \frac{z}{1-z}$$

$$\tilde{R}^{\langle -1\rangle}(z) = \frac{z}{1+z} = \tilde{S}(z)$$

$$S(z) = \frac{1}{z}\tilde{S}(z) = \frac{1}{1+z}$$

If a is R-diagonal

then $\mu_a(B(0,r)) = t$ if

$$\sqrt{S_{a^*a}(t-1)} = \frac{1}{r}$$

$$r = \frac{1}{\sqrt{S_{a^*a}(t-1)}} = \frac{1}{\sqrt{\frac{1}{1+t-1}}} = \frac{1}{\sqrt{\frac{1}{t}}}$$

$$= \sqrt{t} \Rightarrow t = r^2$$

$$\mu_a(B(0,r)) = r^2$$

Lemma 3.3 (in Haagerup's 1997) 7

$$a(s) = (1 - sa)^{-1} - M_a(s)$$

$$b(t) = (1 - tb)^{-1} - M_b(t)$$

$$\varphi((1 - sa)^{-1}) = M_a(s)$$

$$\varphi(a(s)) = \varphi(b(t)) = 0$$

For any $\rho \in \mathbb{C}$

$$(1 - sa)(1 - \rho\, a(s)\, b(t))(1 - tb)$$

$$= c_0 + c_1 a + c_2 b + c_3 ab$$

with
$$c_0 = 1 - \rho \tilde{M}_a(s) \tilde{M}_b(t)$$

$$c_1 = -s(1 - \rho M_a(s) \tilde{M}_b(t))$$

$$c_2 = -t(1 - \rho \tilde{M}_a(s) M_b(t))$$

$$c_3 = st(1 - \rho M_a(s) M_b(t))$$

⑧

Theorem Suppose a & b are free, $\varphi(a) \neq 0$, $\varphi(b) \neq 0$

$$S_a(z) = \frac{1+z}{z} \tilde{M}_a^{\langle -1 \rangle}(z). \text{ Then}$$

$$S_{ab}(z) = S_a(z) S_b(z)$$

Proof: We must show:

$$z \tilde{M}_{ab}^{\langle -1 \rangle}(z) = (1+z) \tilde{M}_a^{\langle -1 \rangle}(z) \tilde{M}_b^{\langle -1 \rangle}(z).$$

So we let $s = \tilde{M}_a^{\langle -1 \rangle}(z)$, $t = \tilde{M}_b^{\langle -1 \rangle}(z)$

let $v = \frac{1+z}{z} st$. We will show

$$\tilde{M}_{ab}(v) = \tilde{M}_a(s) = \tilde{M}_b(t) = z$$

Let $\rho = \dfrac{1}{M_a(s) \tilde{M}_a(s)}$.

$$C_0 = 1 - \rho \tilde{M}_a(s) \tilde{M}_b(t) = \frac{1}{1+z} = \frac{1}{M_b(t)}$$

$$C_3 = st(1 - \rho M_a(s) M_b(t)) = \frac{-st}{\tilde{M}_a(s)}$$

$C_2 = C_1 = 0 \leftarrow$ this how we choose ρ.

$$(1-st)(1-\rho a(s) b(t))(1-tb)$$

$$= \frac{1}{M_a(s)} - \frac{st}{\tilde{M}_a(s)} ab$$

$$= \frac{1}{M_a(s)} \left(1 - \frac{M_a(s)}{\tilde{M}_a(s)} st ab\right)$$

$$= \frac{1}{M_a(s)} (1 - vab)$$

$$(1-tb)^{-1} (1-\rho a(s) b(t))^{-1} (1-sa)^{-1}$$

$$= M_a(s) (1-vab)^{-1}.$$ For $n \geq 1$

$$\varphi\left((M_b(t)+b(t)) (\rho a(s) b(t))^n (M_a(s)+a(s)) \right)$$

$$= 0.$$

Hence

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$$\varphi\left(M_a(s)(1-vab)^{-1}\right)$$

$$= \varphi\left((1-tb)^{-1}(1-\rho\, a(s)\, b(t))^{-1}(1-sa)^{-1}\right)$$

$$= \varphi\left(M_b(t)\, M_a(s)\right)$$

$$= M_a(s)\; M_b(t)$$

Thus $M_{ab}(v) = M_a(s) = M_b(t)$

$$\tilde{M}_{ab}(v) = \tilde{M}_a(s) = \tilde{M}_b(t) = z$$

$$\tilde{M}_{ab}^{\langle -1\rangle}(z) = v = \frac{1+z}{z} \cdot s \cdot t =$$

$$= \frac{1+z}{z}\, \tilde{M}_a^{\langle -1\rangle}(z) \cdot \tilde{M}_b^{\langle -1\rangle}(z)$$

$$\frac{1+z}{z}\, \tilde{M}_{ab}^{\langle -1\rangle}(z) = \frac{1+z}{z}\tilde{M}_a^{\langle -1\rangle}(z) \cdot \frac{1+z}{z}\, \tilde{M}_b^{\langle -1\rangle}(z)$$

$$S_{ab}(z) = S_a(z)\;\; S_b(z) \qquad \square$$

Ex If p & q are free projections

$$S_p(z) = \frac{1+z}{\alpha+z} \qquad \alpha = \varphi(p)$$

$$S_q(z) = \frac{1+z}{\beta+z} \qquad \beta = \varphi(q)$$

$$S_{pq}(z) = \frac{(1+z)^2}{(\alpha+z)(\beta+z)}$$

$\Rightarrow \dots \Rightarrow$ spectral density of pqp is

$$c_0 \delta_0 + \frac{\sqrt{(b-t)(t-a)}}{2\pi t(1-t)} + c_1 \delta_1$$

on $[a,b] \subseteq [0,1]$

$x = \sqrt{(1-\alpha)(1-\beta)}$ $\quad y = \sqrt{\alpha\beta}$

$0 \leq a = 1-(x+y)^2 < b = 1-(x-y)^2 \leq 1$

$c_0 = 1 - \min\{\alpha, \beta\}$

$c_1 = \max\{\alpha+\beta-1, 0\}$

Let c_1 & c_2 be two

Marchenko-Pasturs of paramaters

r & s $\quad c_1$ & c_2 free

$$(c_1+c_2)^{-1/2} c_1 (c_1+c_2)^{-1/2}$$