The Brown Measure of an R-diagonal Operator

25 June 2021

Part I

$\lambda > 0$

①

$x = x^*$ is even: $\varphi(x^{2n-1}) = 0 \quad \forall n$

$b = b^*$ is Bernoulli, $\varphi(b) = 0$, $b^2 = 1$

$$x(s) = (1 - sx)^{-1} - \varphi((1-sx)^{-1})$$
$$= (1 - sx)^{-1} - M_x(s)$$

$$\boxed{\begin{aligned} &\tilde{M}_x(s) \\ &= M_x(s) - 1 \\ &\tilde{M}_{\lambda b}(t) \\ &= M_{\lambda b}(t) - 1 \end{aligned}}$$

$$b(t) = (1 - t\lambda b)^{-1} - \varphi((1 - t\lambda b)^{-1})$$
$$= (1 - t\lambda b)^{-1} - M_{\lambda b}(t)$$

$\rho \in \mathbb{C}$

$$(1 - sx)(1 - \rho\, x(s)\, b(t))(1 - t\lambda b)$$
$$= C_0 + C_1 x + C_2 \lambda b + C_3 \lambda x b$$

$$C_0 = 1 - \rho\, \tilde{M}_x(s)\, \tilde{M}_{\lambda b}(t)$$
$$C_1 = -s(1 - \rho\, M_x(s)\, \tilde{M}_{\lambda b}(t))$$

$C_2 = -t(1-\rho \tilde{M}_x(s) \, M_{\lambda b}(t))$ (2)

$C_3 = st(1 - \rho M_x(s) \, M_{\lambda b}(t))$

Now we want to choose s, t, ρ such that $C_0 = C_3 = 0$ and $C_1 = C_2$. For this to work we must have

$M_x(s) \, M_{\lambda b}(t) = \tilde{M}_x(s) \, \hat{M}_{\lambda b}(t)$ and $\rho = (M_x(s) \, M_{\lambda b}(t))^{-1}$. The condition $M_x(s) \, M_{\lambda b}(t) = \hat{M}_x(s) \, \hat{M}_{\lambda b}(t)$ implies $M_x(s) + M_{\lambda b}(t) = 1$, $M_{\lambda b}(t) = \dfrac{1}{1-\lambda^2 t^2}$ because $\varphi(b^n) = 1$ for all $n \geq 1$.

For $v > 0$, $M_x(iv) = \varphi((1-ivx)^{-1})$

$= \varphi((1+ivx)^{-1}) = \overline{\varphi((1-ivx)^{-1})}$

$= \mathrm{Re}\left(\int (1-ivw)^{-1} \, d\mu_x(w)\right)$

$= \int (1+v^2w^2)^{-1}\, d\mu_x(w)$ ③

Let $s = iv$ and $t = \dfrac{i}{\lambda^2 v}$. Thus

$$M_{\lambda b}(t) = \frac{1}{1-\lambda^2\left(\frac{i}{\lambda^2 v}\right)^2} = \frac{\lambda^2 v^2}{1+\lambda^2 v^2}$$

If we can choose $v > 0$ such that $\boxed{\lambda^2 v^2 = \dfrac{1-M_x(iv)}{M_x(iv)}}$ then

$M_x(s) + M_{\lambda b}(t) = 1$ and $\boxed{s M_x(s) = t M_{\lambda b}(t)}$

$\left[\!\left[\, s = iv \quad t = \dfrac{i}{\lambda^2 v} \,\right]\!\right]$

So now suppose $\lambda^2 v^2 = \dfrac{1-M_x(iv)}{M_x(iv)}$

$= \dfrac{1-M_x(s)}{M_x(s)}$. Observe that

$$\frac{s}{t} = \frac{iv}{i/(\lambda^2 v)} = \lambda^2 v^2 = \frac{1-M_x(iv)}{M_x(iv)} = \frac{M_{\lambda b}(t)}{M_x(s)}$$

$$\lambda^2 v^2 = \frac{1 - M_x(iv)}{M_x(iv)}, \quad 1 + \lambda^2 v^2 = \frac{1}{M_x(iv)}$$

$$\frac{\lambda^2 v^2}{1 + \lambda^2 v^2} = \frac{1 - M_x(iv)}{M_x(iv)} \cdot \frac{M_x(iv)}{1} = 1 - M_x(iv).$$

Thus $M_{\lambda b}(t) = 1 - M_x(s)$ or

$$\boxed{M_x(s) + M_{\lambda b}(t) = 1}$$

Recall $c_1 = -s\left(1 - \rho\, M_x(s)\, \hat{M}_{\lambda b}(t)\right)$

$\rho = \dfrac{1}{M_x(s)\, M_{\lambda b}(t)}$, so

$$c_1 = \frac{-s}{\cancel{M_x(s)}\, M_{\lambda b}(t)} \left[\cancel{M_x(s)}\, M_{\lambda b}(t) - \cancel{M_x(s)}\, \widehat{M_{\lambda b}}(t)\right]$$

$$= \frac{-s}{M_{\lambda b}(t)}$$

$$c_2 = \frac{-t}{M_x(s)} = c_1$$

Hence $(1-sx)(1-\rho x(s)b(t))(1-t\lambda b)$ 5

$$= c_1(x+\lambda b)$$

Main Calculation

$a = bx$ is an R-diagonal operator. If $\theta \in \mathbb{R}$ $e^{i\theta}a$ is also R-diag. and $(e^{i\theta}a)^*(e^{i\theta}a) = a^*a$

$$L(a-\lambda) = L(a+\lambda) = L(bx+\lambda)$$

$$= L(b(x+\lambda b)) = L(b) + L(x+\lambda b)$$

$$= L(c_1^{-1}(1-sx)(1-\rho x(s)b(t))(1-t\lambda b))$$

$$= L(c_1^{-1}) + L(1-sx) + L(1-\rho x(s)b(t)) + L(1-t\lambda b)$$

$$L(c_1^{-1}) = -\log|c_1| = -\log(v) + \log\left(\frac{\lambda^2 v^2}{1+\lambda^2 v^2}\right)$$

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$$L(1-sx) = \int \log|1-sw|\, d\mu_x(w)$$

$$= \frac{1}{2}\int \log|1-sw|^2\, d\mu_x(w)$$

$$= \frac{1}{2}\int \log(1+v^2w^2)\, d\mu_x(w)$$

$s = iv$

$t = \dfrac{i}{\lambda^2 v}$

$$L(1-t\lambda b) = \frac{1}{2}L\big((1-t\lambda b)(1-\bar{t}\lambda b)\big)$$

$$= \frac{1}{2}\log\left(\frac{1+\lambda^2v^2}{\lambda^2v^2}\right)$$

$$L(a-\lambda) = \frac{1}{2}\int \log(1+v^2w^2)\, d\mu_x(w)$$

$$+\frac{1}{2}\log\left(\frac{1+\lambda^2v^2}{\lambda^2v^2}\right) - \log(v) + \log\left(\frac{\lambda^2v^2}{1+\lambda^2v^2}\right)$$

$$+L\big(1-\rho x(s)\, b(t)\big)$$

$$= \frac{1}{2}\int \log(1+v^2w^2)\, d\mu_x(w) + \frac{1}{2}\log\left(\frac{\lambda^2}{1+\lambda^2v^2}\right)$$

$$+L\big(1-\rho x(s)\, b(t)\big)$$

Next week! Freeness

$$\Rightarrow L(1 - \rho\, x(s)\, b(t)) = 0$$

$$0 \leq L(1-\rho\, x(s)\, b(t)) \leq \log(1+\epsilon)$$

for all $\epsilon > 0$.