Freeness and R-diagonal Operators

4 June 2021

$(\mathcal{A}, \varphi)$ = non-commutative probability space

$\mathcal{A}$ = unital algebra over $\mathbb{C}$

$\varphi : \mathcal{A} \longrightarrow \mathbb{C}$, linear, $\varphi(1) = 1$

$\mathcal{A}_1, \ldots, \mathcal{A}_s \subseteq \mathcal{A}$ unital subalgebras are freely independent (w.r.t. φ) if whenever $a_1, \ldots, a_n \in \mathcal{A}$ s.t.

- $\varphi(a_i) = 0$
- $a_i \in \mathcal{A}_{j_i}$ $\quad j_1 \neq j_2, \ldots, j_{n-1} \neq j_n$

then $\varphi(a_1 \cdots a_n) = 0$.

$\mathcal{A} = \mathbb{C}[\mathbb{F}_n] \qquad \mathbb{F}_n = \langle \gamma_1, \ldots, \gamma_n \rangle$

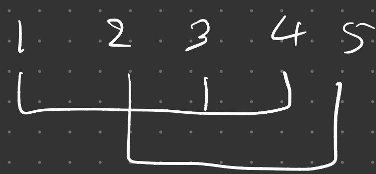
$$f * g(\pi) = \sum_{\sigma \in \mathbb{F}_n} f(\sigma)\, g(\sigma^{-1}\pi)$$

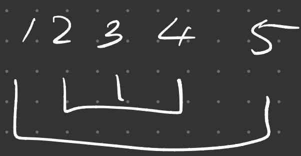
$\mathcal{A}_i = alg(1, \gamma_i, \gamma_i^{-1})$, $\varphi(f) = f(e)$.

$\mathcal{A}_1, \dots, \mathcal{A}_n$ are free.

Partitions and Cumulants

$P(n) = \{$ partitions of $[n]\}$, $[n] = \{1, 2, \dots n\}$





crossing

non-crossing

$NC(n) = \{$ non-crossing partitions $\}$

$\pi \le \sigma$ means every block of π is contained in some block of σ. 1_n = partition with 1 block. $\pi \vee \sigma$ = smallest partition larger than or equal to π and σ.

$a_1, \dots, a_5 \in (\mathcal{A}, \varphi)$ $\pi =$

Notation

$$\varphi_\pi(a_1, a_2, a_3, a_4, a_5)$$

$$= \varphi(a_1 a_5)\,\varphi(a_2 a_4)\,\varphi(a_3)$$

$$K_1(a_1) = \varphi(a_1)$$

$$K_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$$

$$K_3(a_1, a_2, a_3) = \varphi(a_1 a_2 a_3) - \{\varphi(a_1)\varphi(a_2 a_3) + \varphi(a_2)\varphi(a_1 a_3) + \varphi(a_1 a_2)\varphi(a_3)\} + 2\,\varphi(a_1)\varphi(a_2)\varphi(a_3)$$

In general $K_n(a_1, a_2, \dots a_n)$

$$= \sum_{\pi \in NC(n)} \mu(\pi, 1_n)\, \varphi_\pi(a_1, \dots a_n).$$

μ Möbius fn of $NC(n)$.

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} K_\pi(a_1, \dots, a_n)$$

$$K_\pi(a_1, \dots a_n) = \prod_{\substack{V \in \pi \\ V = (i_1 \dots i_j)}} K_j(a_{i_1}, \dots, a_{i_j})$$

$a_1, \dots, a_s \in (\mathcal{A}, \varphi)$ are free if

$\mathcal{A}_i = \text{alg}(1, a_i)$ are free.

Basic Fact $a_1, \dots, a_s \in (\mathcal{A}, \varphi)$ are free if and only "mixed cumulants vanish":

$$K_n(a_{i_1}, \dots, a_{i_n}) = 0 \quad \text{unless}$$

$$i_1 = i_2 = \cdots = i_n$$

Basic Example If x is semi-circular: $\varphi(x^{2n-1}) = 0$ and $\varphi(x^{2n}) = C_n = \frac{1}{n+1}\binom{2n}{n}$.

then $\kappa_n(x, \dots, x) = 0$

except $\kappa_2(x, x) = 1$.

Product Rule

Given $r \geq 2$, $n = n_1 + \dots + n_r$, and $a_1, \dots, a_n \in \mathcal{A}$. Then

$$\kappa_r(a_1 \cdots a_{n_1}, a_{n_1+1} \cdots a_{n_1+n_2}, \dots, a_{n_1+\dots+n_{r-1}+1} \cdots a_{n_1+\dots+n_r})$$

$$= \sum_{\substack{\pi \in NC(n) \\ \pi \vee \rho = 1_n}} \kappa_\pi(a_1, \dots, a_n)$$

$$\rho = \{(1, \dots, n_1), (n_1+1, \dots, n_1+n_2), \dots, (n_1+\dots+n_{r-1}+1, \dots, n_1+\dots+n_r)\}$$

Example $\kappa_{n-1}(a_1 a_2, a_3, \dots, a_n)$

$$= \sum_{\substack{\pi \in NC(n) \\ \pi \vee \{(1,2),(3),\dots,(n)\} = 1_n}} \kappa_\pi(a_1, a_2, \dots, a_n)$$

$\pi \vee \sqcup 1 \cdots 1 = 1_n$ means: either

- π has one block ie $\pi = 1_n$ or

π has 2 blocks

$K_{n-1}(a_1 a_2, a_3, \dots a_n)$

$$= K_n(a_1, \dots, a_n) + \sum_{j=1}^{n-1} K_{n-j}(a_1, a_{j+2}, \dots, a_n)\, K_j(a_2, \dots a_{j+1})$$

Bernoulli Random Variable

$(\mathcal{A}, \varphi) =$ $*$-non-commutative probability space $\varphi(a^*) = \overline{\varphi(a)}$

$\varphi(a^* a) \geq 0$.

b is Bernoulli if $b = b^*$, $b^2 = 1$

$\varphi(b) = 0$.

Find $K_n(b, \dots, b)$ for $n = 1, 2, 3, \dots$ (7)

$$\varphi(b^{2n-1}) = \varphi(b) = 0$$

$$\varphi(b^{2n}) = 1$$

Fact

$K_n(a_1, a_2, \dots, a_n) = 0$ if $a_i = 1$

$$K_{n-1}(b^2, b, \dots, b) = K_{n-1}(1, b, \dots b) = 0$$

$$= K_n(b, \dots b)$$

$$+ \sum_{j=1}^{n-1} K_{n-j}(b, \dots b) K_j(b, \dots b) \quad (\mathcal{A})$$

Let $\alpha_n = (-1)^n K_{2n+2}(b, \dots b)$

$\alpha_1 = -1 \; K_4(b, b, b, b) = 1 \qquad \alpha_0 = 1$

$$1 = \varphi(b^4) = \sum_{\pi} K_\pi(b, b, b, b)$$

$$= K_{\sqcup\!\sqcup\!\sqcup} + K_{\sqcup\!\sqcup} + K_{\sqcup\sqcup}$$

$$= K_4 + 2 K_2^2$$

$\left\{ \begin{array}{l} K_4 = 1 - 2K_2^2 = -1 \\ K_2 = \varphi(b^2) - \varphi(b)^2 = 1 \end{array} \right.$

$$\alpha_{m-1} = \sum_{\ell=0}^{m-1} \alpha_\ell \, \alpha_{m-\ell-1}$$

which is the recursion for the Catalan numbers: $\alpha_n = C_n$

Conclusion $K_n(b, \dots b) = 0$ for n odd

$$K_{2m}(b, \dots, b) = (-1)^{m-1} C_{m-1}$$

Definition $(\mathcal{A}, \varphi)$ $*$-non-commutative probability space. If $a \in \mathcal{A}$ let $a^{(1)} = a$, $a^{(-1)} = a^*$.

If for every $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ we have $K_n(a^{(\varepsilon_1)}, a^{(\varepsilon_2)}, \dots, a^{(\varepsilon_n)}) = 0$

unless
- n is even
- $\varepsilon_i = -\varepsilon_{i+1}$ $\quad 1 \leq i \leq n-1$

then we say a is R-diagonal. 9

Example x_1, x_2 free & semi-circular

$c = \frac{1}{\sqrt{2}}(x_1 + i x_2)$ is R-diagonal

$$c^{(\varepsilon)} = \frac{1}{\sqrt{2}}(x_1 + i\varepsilon x_2)$$

$$K_n(c^{(\varepsilon_1)}, \dots, c^{(\varepsilon_n)})$$

$$= 2^{-n/2} K_n(x_1 + i\varepsilon_1 x_2, \dots, x_1 + i\varepsilon_n x_2)$$

$$= 2^{-n/2}\{ K_n(x_1, \dots x_1) + (i)^n \varepsilon_1 \cdots \varepsilon_n K_n(x_2, \dots x_2) \}$$

$= 0$ unless $n=2$. when $n=2$

$$2^{-2/2}(K_2(x_1, x_1) - \varepsilon_1\varepsilon_2 K_2(x_2, x_2)\} = \frac{1-\varepsilon_1\varepsilon_2}{2}$$

$= 0$ unless $\varepsilon_1 \neq \varepsilon_2$

$$\kappa_n(c^{(\varepsilon_1)}, \ldots, c^{(\varepsilon_n)}) = \begin{cases} 1 & \varepsilon_1 = -\varepsilon_2 \\ 0 & n \neq 2 \end{cases}$$