# Seminar on Brown Measure ①

25 March 2021

◇

## §1 Brown Measure for Matrices

$T \in M_N(\mathbb{C})$  $\mathrm{Tr}(T) = \sum_{i=1}^{N} t_{ii}$

$\lambda_1, \ldots, \lambda_N$ eigenvalues of $T$

$|T| = \sqrt{T^* T}$.  $g(x,y) = \log\sqrt{x^2+y^2}$

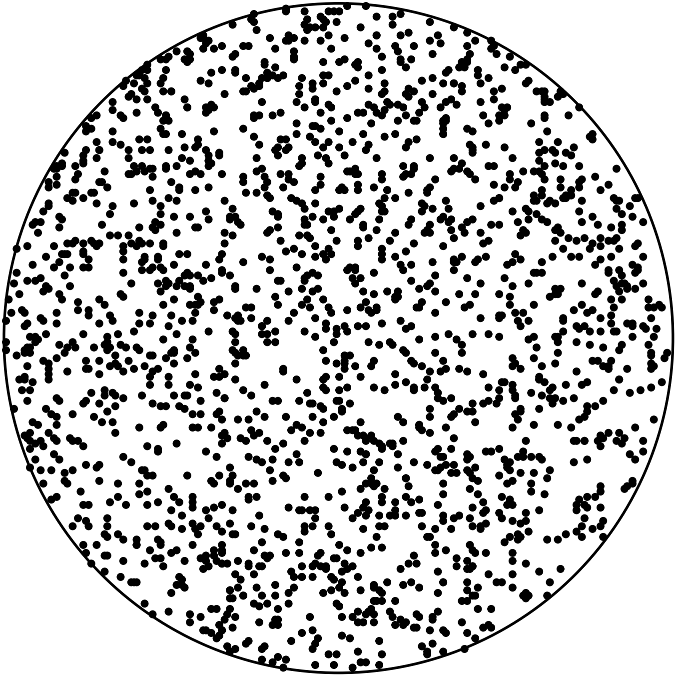
loc. integrable even on nbhd of $(0,0)$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Thm; $f \in C_c^\infty(\mathbb{R}^2)$

$$\int \nabla^2 f(x,y) \log[\det|x+iy-T|]\,dx\,dy$$

$$= 2\pi \sum_{i=1}^{N} f(\lambda_i) \quad \lambda = x+iy$$

Let $\mu_T = \frac{1}{2\pi N} \nabla^2 \log[\det|\lambda - T|]$ in the sense of dist.



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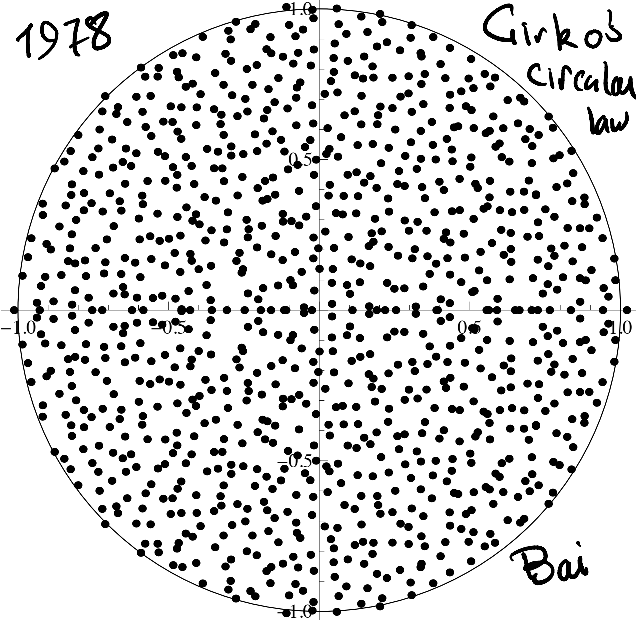
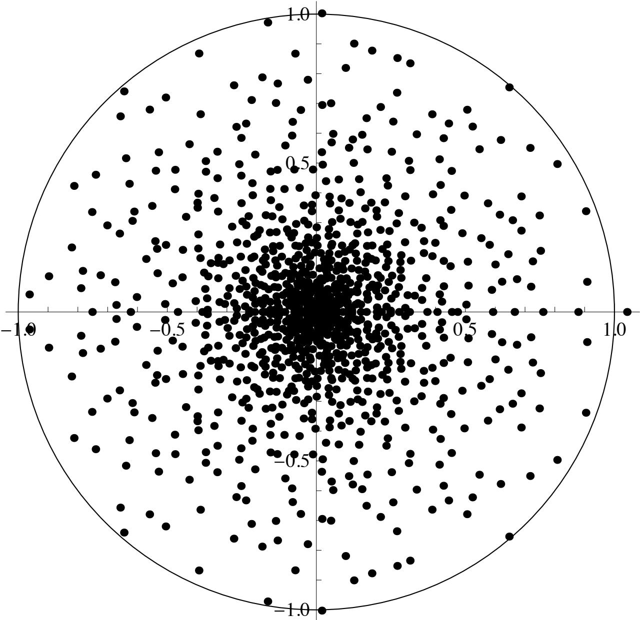
Girko's circular law




Bai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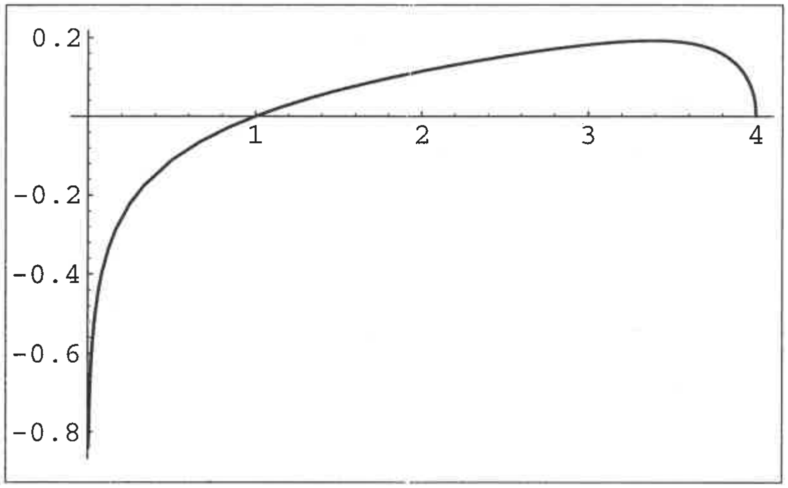


$$f(x) = \log x \frac{\sqrt{4(n-1)-x^2}}{n^2-x^2} \qquad (n=5)$$

# Status Report

for

Flemming Larsen

# Chapter 11
# Brown Measure

The Brown measure is a generalization of the eigenvalue distribution for a general (not necessarily normal) operator in a finite von Neumann algebra (i.e, a von Neumann algebra which possesses a trace). It was introduced by Larry Brown in [46], but fell into obscurity soon after. It was revived by Haagerup and Larsen [85], and played an important role in Haagerup's investigations around the invariant subspace problem [87]. By using a "hermitization" idea one can actually calculate the Brown measure by $M_2(\mathbb{C})$-valued free probability tools. This leads to an extension of the algorithm from the last chapter to the calculation of arbitrary polynomials in free variables. For generic non-self-adjoint random matrix models their asymptotic complex eigenvalue distribution is expected to converge to the Brown measure of the ($*$-distribution) limit operator. However, because the Brown measure is not continuous with respect to convergence in $*$-moments this is an open problem in the general case.

## 11.1 Brown measure for normal operators

Let $(M,\tau)$ be a $W^*$-probability space and consider an operator $a \in M$. The relevant information about $a$ is contained in its $*$-distribution which is by definition the collection of all $*$-moments of $a$ with respect to $\tau$. In the case of self-adjoint or normal $a$ we can identify this distribution with an analytic object, a probability measure $\mu_a$ on the spectrum of $a$. Let us first recall these facts.

If $a = a^*$ is self-adjoint, there exists a uniquely determined probability measure $\mu_a$ on $\mathbb{R}$ such that for all $n \in \mathbb{N}$

$$\tau(a^n) = \int_{\mathbb{R}} t^n d\mu_a(t)$$

and the support of $\mu_a$ is the spectrum of $a$; see also the discussion after equation (2.2) in Chapter 2.

More general, if $a \in M$ is *normal* (i.e., $aa^* = a^*a$), then the spectral theorem provides us with a projection valued spectral measure $E_a$ and the Brown measure is just the spectral measure $\mu_a = \tau \circ E_a$. Note that in the normal case $\mu_a$ may not be determined by the moments of $a$. Indeed, if $a = u$ is a Haar unitary then the moments of $u$ are the same as the moments of the zero operator. Of course, their $*$-moments are different. For a normal operator $a$ its spectral measure $\mu_a$ is uniquely determined by

$$\tau(a^n a^{*m}) = \int_{\mathbb{C}} z^n \bar{z}^m d\mu_a(z) \tag{11.1}$$

for all $m, n \in \mathbb{N}$. The support of $\mu_a$ is again the spectrum of $a$.

We will now try to assign to any operator $a \in M$ a probability measure $\mu_a$ on its spectrum, which contains relevant information about the $*$-distribution of $a$. This $\mu_a$ will be called the *Brown measure* of $a$. One should note that for non-normal operators there are many more $*$-moments of $a$ than those appearing in (11.1). There is no possibility to capture all the $*$-moments of $a$ by the $*$-moments of a probability measure. Hence, we will necessarily loose some information about the $*$-distribution of $a$ when we go over to the Brown measure of $a$. It will also turn out that we need our state $\tau$ to be a trace in order to define $\mu_a$. Hence in the following we will only work in tracial $W^*$-probability spaces $(M, \tau)$. Recall that this means that $\tau$ is a faithful and normal trace. Von Neumann algebras which admit such faithful and normal traces are usually addressed as *finite* von Neumann algebras. If $M$ is a finite factor, then a tracial state $\tau : M \to \mathbb{C}$ is unique on $M$ and is automatically normal and faithful.

## 11.2 Brown measure for matrices

In the finite-dimensional case $M = M_n(\mathbb{C})$, the Brown measure $\mu_T$ for a normal matrix $T \in M_n(\mathbb{C})$, determined by (11.1), really is the eigenvalue distribution of the matrix. It is clear that in the case of matrices we can extend this definition to the general, non-normal case. For a general matrix $T \in M_n(\mathbb{C})$, the spectrum $\sigma(T)$ is given by the roots of the characteristic polynomial

$$P(\lambda) = \det(\lambda I - T) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n),$$

where $\lambda_1, \dots, \lambda_n$ are the roots repeated according to algebraic multiplicity. In this case we have as eigenvalue distribution (and thus as Brown measure)

$$\mu_T = \frac{1}{n}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_n}).$$

We want to extend this definition of $\mu_T$ to an infinite dimensional situation. Since the characteristic polynomial does not make sense in such a situation we have to find an analytic way of determining the roots of $P(\lambda)$ which survives also in an infinite dimensional setting.

Consider

$$\log |P(\lambda)| = \log |\det(\lambda I - T)| = \sum_{i=1}^{n} \log |\lambda - \lambda_i|.$$

CAHIERS SCIENTIFIQUES
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FASCICULE XXV

# LES ALGÈBRES D'OPÉRATEURS DANS L'ESPACE HILBERTIEN

## (ALGÈBRES DE VON NEUMANN)

PAR

**Jacques DIXMIER**
PROFESSEUR A LA FACULTÉ DES SCIENCES DE PARIS

*DEUXIÈME ÉDITION*
*revue et augmentée* (1st ed 1957)

PARIS
GAUTHIER-VILLARS ÉDITEUR
1969

$$\mu_T = \frac{1}{2\pi N} \nabla^2 \log|\det[x+iy-T]|$$ ②

$$= \frac{1}{2\pi} \nabla^2 \operatorname{tr}(\log|x+iy-T|)$$

$$\operatorname{tr}(T) = \frac{1}{N}\operatorname{Tr}(T)$$

## §2 Extensions to von Neumann

$H$ = sep. Hilbert space, $B(H)$

= bdd. linear ops. $H \to H$.

$\xi, \eta \in H$ $\quad \{T_n\}_n \subseteq B(H)$

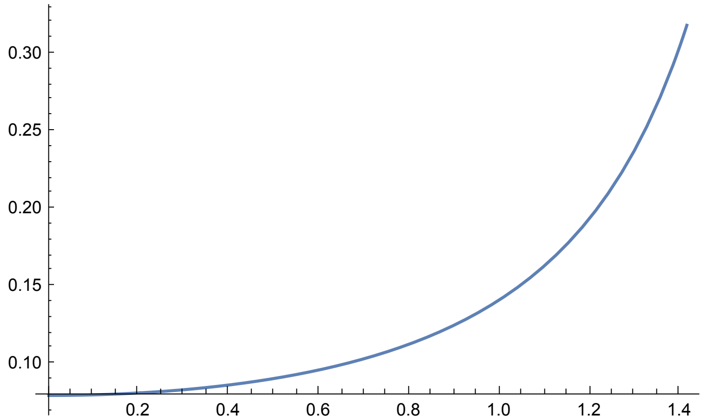
$T_n \xrightarrow{WOT} T$ if $\forall \xi, \eta$

$$\langle (T_n - T)\xi, \eta \rangle \longrightarrow 0$$

$M \subseteq B(H)$ selfadjoint, contains $\mathbb{1}$, WOT closed sub-algebra = von Neumann alg.

In[8]:= Plot$\left[\frac{4}{\pi\left(4-r^2\right)^2},\ \left\{r,\ 0,\ \sqrt{2}\right\}\right]$

Out[8]=




In[10]:= $2\pi\,$Integrate$\left[r\ \frac{4}{\pi\left(4-r^2\right)^2},\ \left\{r,\ 0,\ \sqrt{2}\right\}\right]$

Out[10]= 1

$M \subseteq B(H)$ v.N. alg.

$\tau : M \to \mathbb{C}$, linear

- $\tau(ab) = \tau(ba) \quad \forall a, b \in M$
- $\tau(x^*x) \geq 0 \quad \forall x \in M$
- $\tau(x^*x) = 0 \iff x = 0$
- $\tau(1) = 1$
- WOT cont.: $x_n \xrightarrow{WOT} x$

$\Rightarrow \tau(x_n) \to \tau(x)$

$(M_N(\mathbb{C}), \mathrm{tr})$, $(L^\infty(\mu), \int)$

$M_2 \to M_2 \otimes M_2 \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$

$\overline{\bigotimes_{n=1}^{\infty} M_2}^{WOT} = M_{2^\infty}$

## §3 Examples

$s_1, s_2$ free & semi-circular

$c = \frac{1}{\sqrt{2}}(s_1 + i s_2)$ Voiculescu's circular op.

$\mu_c$ = uniform measure on $\pi$

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## Proof of Thm

$f \in C_c^\infty(\mathbb{R}^2)$ claim:

$$\int \nabla^2 f(x,y) \log \sqrt{x^2+y^2}\, dx\, dy = 2\pi f(0,0)$$

Choose $R>0$ $\operatorname{supp}(f) \subseteq D_R$ open

$D_{r,R} = \{(x,y) \mid r < \sqrt{x^2+y^2} < R\}$

Calculus lemma: $\nabla \cdot (f \nabla g - g \nabla f)$

$$= f \nabla^2 g - g \nabla f$$

$g(x,y) = \log \sqrt{x^2+y^2}$ $\quad \nabla^2 g = 0$

$$\nabla g = \left( \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right) \text{ on } \partial D_r$$

let $\vec{n}$ = outward pointing normal

$$= \left( \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

$$\nabla g \cdot \vec{n} = \frac{1}{r} \text{ on } \partial D_r$$

$$\int_{D_{r,R}} \nabla^2 f g \, dx\, dy = \int_{D_{r,R}} \nabla \cdot (g \nabla f - f \nabla g) \, dx\, dy$$

$$= \int_{\partial D_{r,R}} (g \nabla f - f \nabla g) \cdot \vec{n} \, ds$$

$$= \int_{\partial D_{r,R}} g \nabla f \cdot \vec{n}\, ds - \int_{\partial D_{r,R}} f \nabla g \cdot \vec{n}\, ds$$

$$= \frac{1}{r} \int_0^{2\pi} r f(r\cos\theta, r\sin\theta)\, d\theta$$

$$- r \log r \int_0^{2\pi} \frac{\partial f}{\partial r}(r\cos\theta, r\sin\theta)\, d\theta$$

$$\to 2\pi f(0,0) \qquad \blacksquare$$

$$\det(\lambda - T) = \prod_{i=1}^{N} (\lambda - \lambda_i)$$

$$\det|\lambda - T| = |\det(\lambda - T)|$$

$$= \prod_{i=1}^{N} |\lambda - \lambda_i|$$

$$\log(\det|\lambda - T|) = \sum_{i=1}^{N} \log|\lambda - \lambda_i|$$

$$\|$$

$$\mathrm{Tr}\left(\log|\lambda - T|\right)$$

$$\frac{1}{2\pi}\mathrm{tr}\left(\log|\lambda - T|\right) = \frac{1}{2\pi N}\sum_{i=1}^{N}\log|\lambda - \lambda_i|$$

$$\int \nabla^2 f(x,y)\left(\frac{1}{2\pi}\mathrm{tr}\left(\log|\lambda - T|\right)\right)dx\,dy$$

$$= \frac{1}{2\pi N}\sum_{i=1}^{N}\int \nabla f(x,y)\log|\lambda - \lambda_i|\,dx\,dy$$

$$= \frac{1}{N}\sum_{i=1}^{N} f(\lambda_i)$$

$$= \int f(\lambda)\,d\mu_T(\lambda)$$

$$d\mu_T = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$$

$V$ is a vector space

$P: V \longrightarrow V$ is a linear transformation such that $P^2 = P$ i.e. $P(P(v)) = P(v)$, $P$ is a projection.

$\ker(P) = \{v \mid P(v) = 0\}$

$\operatorname{ran}(P) = \{P(v) \mid v \in V\}$

given $v \in V$ $\exists\, w, z \in V$ $w \in \operatorname{ran}(P)$, $z \in \ker(P)$, and

$$v = w + z.$$

$w = P(v)$ $\quad z = (I - P)(v) = v - w$

$$w + z = P(v) + (I - P)(v)$$

$$= (P + I - P)(v) = I(v) = v$$

$$P(z) = P(I - P)(v) = (P - P^2)(v)$$

$$= 0 \Rightarrow z \in \ker(P)$$

$$\underset{\operatorname{ran}(P)}{\underset{\cap}{P(w)}} = P(P(v)) = P(v) = w$$