Notation

$$a_\lambda = a - \lambda I = a - \lambda$$

$$\begin{aligned} L : \mathcal{M} &\longrightarrow \mathbb{R} \cup \{-\infty\} \\ a &\longmapsto \tau(\log|a|) \end{aligned}$$

$$L_\varepsilon : a \longmapsto \frac{1}{2} L(a^*a + \varepsilon)$$

$$\int_{\mathbb{C}} \varphi(\lambda)\, d\mu_a(\lambda) = \frac{1}{2\pi}\int_{\mathbb{C}} \nabla^2 \varphi(\lambda) L(a_\lambda)\, d\lambda$$

$$\int_{\mathbb{C}} \varphi(\lambda)\, d\mu_{a,\varepsilon}(\lambda) = \frac{1}{2\pi}\int_{\mathbb{C}} \nabla^2 \varphi(\lambda) L_\varepsilon(a_\lambda)\, d\lambda$$

$$\mu_{a,\varepsilon} \xrightarrow{w^*} \mu_a$$

Theorem: $d\mu_{a,\varepsilon}$ is a probability, i.e.,

$$\mu_{a,\varepsilon}(\mathbb{C}) = 1.$$

Pf: Set $g_\varepsilon(\lambda) = 2\frac{d}{d\lambda}L_\varepsilon(a_\lambda)$

$$= -\tau\left((a_\lambda^* a_\lambda + \varepsilon)^{-1} a_\lambda^*\right).$$

For p a polynomial

$$\frac{d}{d\lambda}\tau(p(a_\lambda^* a_\lambda)) = -\tau\left(p'(a_\lambda^* a_\lambda)\, a_\lambda^*\right).$$

Since we have a power series representation valid on a neighbourhood of λ and ε (with $\operatorname{Re}(\varepsilon) > 0$) we get this identity for $L_\varepsilon(a_\lambda) = \tau\left(\log|a_\lambda^* a_\lambda + \varepsilon|\right)$ too.

For $|\lambda| > \|a\|$, set $g(\lambda) = -\tau\left((a-\lambda)^{-1}\right)$

$$= \lambda^{-1}\left(\sum_{j=0}^{\infty} \frac{\tau(a^j)}{\lambda^j}\right).$$

For $|\lambda|$ large,

$$\left\| (a_\lambda^* a_\lambda + \varepsilon)^{-1} a_\lambda^* - a_\lambda^{-1} \right\|$$

$$= \left\| \left[\left(1 + \varepsilon (a_\lambda^* a_\lambda)^{-1}\right)^{-1} - 1 \right] a_\lambda^{-1} \right\|$$

$$\leq \left\| \left(1 + \varepsilon (a_\lambda^* a_\lambda)^{-1}\right)^{-1} - 1 \right\| \, \|a_\lambda^{-1}\|$$

$$= \left\| \sum_{n=1}^{\infty} \varepsilon^n (a_\lambda^* a_\lambda)^{-n} \right\| \, \|a_\lambda^{-1}\|$$

$$\leq \sum_{n=1}^{\infty} \varepsilon^n \|(a_\lambda^* a_\lambda)^{-1}\|^n \, \|a_\lambda^{-1}\|$$

$$= \sum_{n=1}^{\infty} \varepsilon^n \|a_\lambda^{-1}\|^{2n+1}$$

$$= \varepsilon \|a_\lambda^{-1}\|^3 \frac{1}{1 - \varepsilon \|a_\lambda^{-1}\|^2} = O\left(\frac{1}{|\lambda|^3}\right)$$

as $|\lambda| \to \infty$

$$\left| g_\varepsilon(\lambda)\lambda - g(\lambda)\lambda \right| \leq \| \cdots \| \, |\lambda| = O\left(\frac{1}{|\lambda|^2}\right).$$

$$2\pi \mu_{a,\varepsilon}(r\mathbb{D})$$

$$= \iint_{r\mathbb{D}} \nabla^2 L_\varepsilon(a_{x+iy})\, dx\, dy$$

$$= \iint_{r\mathbb{D}} \frac{d^2}{dx^2} L_\varepsilon(a_{x+iy}) + \frac{d^2}{dy^2} L_\varepsilon(a_{x+iy})\; dx\, dy$$

$$= \iint_{r\mathbb{D}} \frac{d}{dx} \operatorname{Re} g_\varepsilon - \frac{d}{dy} \operatorname{Im} g_\varepsilon \; dx\, dy$$

$$\underset{\text{Thm}}{=} \int_{r\mathbb{T}} \operatorname{Im} g_\varepsilon \, dx + \operatorname{Re} g_\varepsilon \, dy$$

$$= \int_0^{2\pi} (\operatorname{Im} g_\varepsilon, \operatorname{Re} g_\varepsilon) \cdot (r\cos t, r\sin t)' \, dt$$

$$\underset{\alpha(t) = (r\cos t, r\sin t)}{=} \int_0^{2\pi} r\cos t \operatorname{Re} g_\varepsilon(\alpha(t)) - r\sin t \operatorname{Im} g_\varepsilon(\alpha(t)) \, dt$$

$$= \int_0^{2\pi} \operatorname{Re}\big(\alpha(t) g_\varepsilon(\alpha(t))\big) \, dt$$

$$= \int_0^{2\pi} \operatorname{Re}\big(\alpha(t) g(\alpha(t))\big) dt + O\left(\frac{1}{r^2}\right)$$

as $r \longrightarrow \infty$

$$= \int_0^{2\pi} 1 + \operatorname{Re} \sum_{n=1}^{\infty} \frac{\tau(a^n)}{\alpha(t)^n} \, dt + O\left(\frac{1}{r^2}\right)$$

$$\left(\text{Since } g_\varepsilon(\alpha(t)) = \tau\big((a - \alpha(t))^{-1}\big) \right)$$

$$= 2\pi + \operatorname{Re} \sum_{n=1}^{\infty} \tau(a^n) \int_0^{2\pi} \frac{1}{\alpha(t)^n} dt + O\left(\frac{1}{r^2}\right)$$

$$2\pi\, \mu_{a,\varepsilon}(r\mathbb{D}) = 2\pi + \mathcal{O}\left(\frac{1}{r^2}\right) \longrightarrow 2\pi$$

Hence $\mu_{a,\varepsilon}(\mathbb{C}) = 1.$ $\boxtimes$

$$\delta_n \xrightarrow{w^*} 0 \qquad \delta_n \not\xrightarrow{w} 0$$

For all $\lambda \in \mathbb{C}$

$$\frac{1}{2\pi}\nabla^2 L_\varepsilon(a_\lambda) = \frac{\varepsilon}{\pi}\tau\left((a_\lambda a_\lambda^* + \varepsilon)^{-1}(a_\lambda^* a_\lambda + \varepsilon)^{-1}\right)$$

For $|\lambda|$ large,

$$\leq \frac{\varepsilon}{\pi}\tau\left((a_\lambda a_\lambda^*)^{-1}(a_\lambda^* a_\lambda)^{-1}\right)$$

$$\leq \frac{\varepsilon}{\pi}\left\|(a_\lambda a_\lambda^*)^{-1}(a_\lambda^* a_\lambda)^{-1}\right\|$$

$$\leq \frac{\varepsilon}{\pi}\left\|(a_\lambda a_\lambda^*)^{-1}\right\|\left\|(a_\lambda^* a_\lambda)^{-1}\right\|$$

$$= \frac{\varepsilon}{\pi}\|a_\lambda^{-1}\|^4 = \varepsilon\, \mathcal{O}\left(\frac{1}{|\lambda|^4}\right)$$

as $|\lambda| \longrightarrow \infty.$

Now suppose $\varphi \in C^\infty_{c,+}$ is 1 on $r\mathbb{D}$ and ≤ 1 on $\mathbb{C}$, $r > \|a\|$.

$$\mu_a(\mathbb{C}) = \int_{\mathbb{C}} \varphi(\lambda)\, d\mu_a(\lambda)$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{C}} \varphi(\lambda)\, d\mu_{a,\varepsilon}(\lambda)$$

$$= 1 - \lim_{\varepsilon \to 0} \int_{\mathbb{C}} 1 - \varphi(\lambda)\, d\mu_{a,\varepsilon}(\lambda)$$

$$\geq 1 - \lim_{\varepsilon \to 0} \int_{r\mathbb{D}^c} 1\, d\mu_{a,\varepsilon}$$

$$= 1 - \mu_{a,\varepsilon}(r\mathbb{D}^c)$$

$$\mu_{a,\varepsilon}(r\mathbb{D}^c) = \frac{1}{2\pi} \int_{r\mathbb{D}^c} \nabla^2 L_\varepsilon(a_\lambda)\, d\lambda$$

$$\leq \int_{r\mathbb{D}^c} \varepsilon\, O\left(\frac{1}{|\lambda|^4}\right) d\lambda \lesssim \varepsilon$$

$$\xrightarrow[\varepsilon \to 0]{} 0 .$$

Hence $\mu_a(\mathbb{C}) = 1$.

Theorem:

(i) μ_a is the unique compactly supported measure on $\mathbb{C}$ so that

$$L(a_\lambda) = \int_{\mathbb{C}} \log|t-\lambda| \, d\mu_a(t)$$

for a.e. $\lambda \in \mathbb{C}$.

(ii) Equality in the above statement holds everywhere in $\sigma(a)^c$.

(iii) $\tau(a^n) = \int t^n \, d\mu_a(t)$ for all $n \in \mathbb{N}$.