$$\xrightarrow[\varepsilon \to 0]{} 0 .$$

Hence $\mu_a(\mathbb{C}) = 1$.

Theorem:

(i) μ_a is the unique compactly supported measure on $\mathbb{C}$ so that

$$L(a_\lambda) = \int_{\mathbb{C}} \log|t-\lambda| \, d\mu_a(t)$$

for a.e. $\lambda \in \mathbb{C}$.

(ii) Equality in the above statement holds everywhere in $\sigma(a)^c$.

(iii) $\tau(a^n) = \int t^n \, d\mu_a(t)$ for all $n \in \mathbb{N}$.

Pf:

(i) Set $\mathcal{J} = \log|\cdot| * \mu_a$.

$$\mathcal{J}(\lambda) = \int_{\mathbb{C}} \log|t-\lambda| \, d\mu_a(t)$$

Note $\mathcal{J} \in L^1_{loc}(\mathbb{C})$.

$$\left(\int_K \int_{\mathbb{C}} \big|\log|\lambda - z|\big| \, d\mu_a(z) \overset{\text{(Tonelli)}}{=} \int_{\mathbb{C}} \int_K \big|\log|\lambda - z|\big| \, dz \, d\mu_a(\lambda) \le C(K). \right)$$

Lemma: Sps $g, h : \mathbb{C} \to \mathbb{R}$ are locally integrable, $|g(z) - h(z)| \xrightarrow{|z| \to \infty} 0$, and $\nabla^2 g = \nabla^2 h$ as distributions.

Then $g = h$ a.e.

Pf: $\nabla^2 (g-h) = 0$ so $g-h$

is harmonic. But Weyl's Lemma implies $g-h$ is equal a.e. to a harmonic function, which tends to 0 at ∞. Hence $g-h=0$ a.e.

Note $L(a_\lambda)$ is subharmonic, hence locally integrable.

Suppose $|\lambda| > \|a\|$

$$L(a-\lambda) = \log|\lambda| - \operatorname{Re} \sum_{n \geq 1} \frac{1}{n\lambda^n} \tau(a^n)$$

$$f(\lambda) = \int_{\mathbb{C}} \log|z-\lambda| \, d\mu_a(z)$$

$$= \int_{\mathbb{C}} \log|\lambda| - \operatorname{Re} \sum_{n \geq 1} \frac{1}{n\lambda^n} z^n \, d\mu_a(z)$$

$$= \log|\lambda| - \operatorname{Re} \sum_{n \geq 1} \frac{1}{n\lambda^n} \int_{\mathbb{C}} z^n \, d\mu_a(z)$$

$$\left| L(a-\lambda) - f(\lambda) \right|$$

$$\leq \sum_{n \geq 1} \frac{1}{n \lambda^n} \left| \tau(a^n) - \int_{\mathbb{C}} z^n \, d\mu_a(z) \right|$$

$$\leq 2 \sum_{n \geq 1} \frac{\|a\|^n}{|\lambda|^n} = \frac{2\|a\|}{|\lambda| - \|a\|} \xrightarrow{|\lambda| \to \infty} 0$$

As distributions,

$$\nabla^2 f = \nabla^2 \left(\log|\cdot| * \mu_a \right)$$

$$= \underbrace{\left(\nabla^2 \log|\cdot| \right)} * \mu_a$$

$2\pi \mu_0$, the Brown measure of 0.

$$= \left(2\pi \delta_0 \right) * \mu_a$$

$$= 2\pi \mu_a .$$

$$\nabla^2 L(a_\lambda) = 2\pi \mu_a \quad \text{by definition}$$

Hence $\quad L(a_\lambda) = f(\lambda) \quad$ a.e.

Suppose now that μ is a measure on $\mathbb{C}$ so that $L(a_\lambda) = \int_{\mathbb{C}} \log|z-\lambda|\,d\mu(z)$.

Then $\log|\cdot| * \mu = \log|\cdot| * \mu_a$ a.e. as functions on $\mathbb{C}$.

$$\begin{aligned} 2\pi\mu &= 2\pi\,\delta_0 * \mu \\ &= \left(\nabla^2 \log|\cdot|\right) * \mu \\ &= \left(\nabla^2 \log|\cdot|\right) * \mu_a \\ &= 2\pi\,\mu_a . \end{aligned}$$

Hence $\mu = \mu_a$.

(iii) We saw

$L(a_\lambda)$ and $\mathcal{F}(\lambda)$ have power series

$$L(a_\lambda) = \log|\lambda| - \mathrm{Re}\sum_{n\geq 1}\frac{1}{n\lambda^n}\int z^n\,d\mu_a(z) = \log|\lambda| - \mathrm{Re}\sum_{n\geq 1}\frac{1}{n\lambda^n}\,\tau(a^n)$$

So for all n,

$$\frac{1}{n\lambda^n}\int z^n \, d\mu_a(z) = \frac{1}{n\lambda^n}\tau(a^n).$$

(ii) $\lambda \mapsto L(a_\lambda)$ is harmonic on $\mathbb{C}\setminus\sigma(a)$, in particular, continuous.

Meanwhile,

$$|\mathcal{F}(s)-\mathcal{F}(t)| \le \int_{\mathbb{C}} \big|\log|s-z| - \log|t-z|\big| \, d\mu_a(z)$$

$$\xrightarrow[DCT]{t\to s} 0.$$

Hence $\mathcal{F}$ is cts on $\mathbb{C}\setminus\sigma(a)$.

Since $\mathcal{F}(\lambda) = L(a_\lambda)$ a.e. on $\mathbb{C}\setminus\sigma(a)$, they must agree everywhere on $\mathbb{C}\setminus\sigma(a)$. $\square$

Theorem: Suppose $a \in \mathcal{M}$. Then:

(i) for any polynomial p,

$$\mu_{p(a)} = \mu_a \# p$$

$$\left(\mu_{p(a)}(A) = \mu_a\left(p^{-1}(A)\right) \right)$$

(ii)

$$\mu_{a^*} = \mu_a \# \bar{\cdot}$$

(iii) if a is invertible

$$\mu_{a^{-1}} = \mu_a \# \cdot^{-1}$$

Pf: Suppose p is a (non-constant) polynomial and $\lambda \in \mathbb{C}$.

Take $\alpha, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$ so that

$$p(a) - \lambda = \alpha (a - \lambda_1)(a - \lambda_2) \cdots (a - \lambda_n).$$

Then

$$\int_{\mathbb{C}} \log|z-\lambda|\, d\mu_{p(a)}(z) = L\big(p(a)-\lambda\big)$$

$$= L\big(\alpha(a-\lambda_1)\cdots(a-\lambda_n)\big)$$

$$= L(\alpha) + L(a-\lambda_1) + \cdots + L(a-\lambda_n)$$

$$= \log|\alpha| + \sum_{j=1}^{n} \int_{\mathbb{C}} \log|z-\lambda_j|\, d\mu_a(z)$$

$$= \int_{\mathbb{C}} \log\big|\alpha(z-\lambda_1)\cdots(z-\lambda_n)\big|\, d\mu_a(z)$$

$$= \int_{\mathbb{C}} \log|p(z)-\lambda|\, d\mu_a(z)$$

$$= \int_{\mathbb{C}} \log|y-\lambda|\, d(\mu_a \# p)(y).$$

a.e.

Hence $\mu_{p(a)} = \mu_a \# p$. $\square$