Hence $\mu_a(C) = 1$.

Theorem:

(i) $\mu_a$ is the unique compactly supported measure on $C$ so that

$$L(\alpha) = \int \log|t - \lambda| \, d\mu_a(t)$$

for $\alpha \in C$.

(ii) Equality in the above statement holds everywhere in $\sigma(a)^c$.

(iii) $\tau(a^n) = \int t^n \, d\mu_a(t)$ for all $n \in \mathbb{N}$. 
Define: Set \( f = \log |t| \ast \mu_a \).

\[ f(x) = \int_C \log |t-x| \, d\mu_a(t) \]

Note \( f \in L^1_{loc}(\mathbb{C}) \).

\[
\left( \int_{\mathbb{C}} |\log |x-z|| \, d\mu_a(z) \right) \leq C(K).
\]

Lemma:

Sps \( g,h : \mathbb{C} \to \mathbb{R} \) are locally integrable, \( |g(z)-h(z)| \xrightarrow{|z| \to \infty} 0 \), and \( \nabla^2 g = \nabla^2 h \) as distributions.

Then \( g = h \) a.e.

So: \( \nabla^2 (g-h) = 0 \) so \( g-h \)
is harmonic. But Weyl’s Lemma implies $g - h$ is equal a.e. to a harmonic function, which tends to 0 at $\infty$. Hence $g - h = 0$ a.e.

Note $L(\alpha, \lambda)$ is subharmonic, hence locally integrable.

Suppose $|\lambda| > |\alpha|$. Then

$$L(\alpha - \lambda) = \log |\lambda| - \Re \sum_{n \geq 1} \frac{1}{n \lambda^n} Z(\alpha^n)$$

$$f(\lambda) = \int \log |z - \lambda| \, d\mu_a(z)$$

$$= \int \log |\lambda| - \Re \sum_{n \geq 1} \frac{1}{n \lambda^n} \sum \int_{\mathbb{C}} z^n \, d\mu_a(z)$$
$$\left| L(a-x) - f(x) \right|$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^x} \left| \zeta(a^n) - \int_C z^n \, d\mu_a(z) \right|$$

$$\leq 2 \sum_{n=1}^{\infty} \frac{\|a\|^n}{|x|^n} = \frac{2\|a\|}{|x| - \|a\|} \xrightarrow{|x| \to \infty} 0$$

As distributions,

$$\nabla^2 y \log 1 - \mu_a$$

$$= \left( \nabla^2 \log 1 \right) \ast \mu_a$$

$$= \left( 2\pi \delta_0 \right) \ast \mu_a$$

$$= 2\pi \mu_a$$

$$\nabla^2 L(a_x) = 2\pi \mu_a$$ by definition

Hence $$L(a_x) = f(x)$$ a.e.
Suppose now that \( \mu \) is a measure on \( C \) so that

\[
L(\alpha) = \int_C \log|z - \lambda| \, d\mu(z).
\]

Then \( \log|1| * \mu = \log|1| * \mu_a \) a.e. as functions on \( C \).

\[
2\pi \mu = 2\pi \delta_0 * \mu = (\nabla^2 \log|1|) * \mu = (\nabla^2 \log|1|) * \mu_a = 2\pi \mu_a.
\]

Hence \( \mu = \mu_a \).

(\text{iii}) We saw

\[
L(\alpha) \text{ and } f(x) \text{ have power series }
\]

\[
\log|1| = \sum_{n \geq 1} \frac{1}{n!} \int \frac{1}{z^n} \, d\mu_a(z) = \log|1| - \text{Re} \sum_{n \geq 1} \frac{1}{n!} \alpha^n.
\]
So for all $n$,
\[
\frac{1}{n^a} \int z^n \, d\mu_a(z) = \frac{1}{n^a} \zeta(a^k).
\]

(iii) $\lambda \mapsto \mathbb{L}(\alpha)$ is harmonic on $C \setminus \sigma(a)$, in particular, continuous.

Meanwhile,
\[
|\mathcal{F}(s) - \mathcal{F}(t)| \leq \int \left| \log |s - z| - \log |t - z| \right| \, d\mu_a(z).
\]

\[
\lim_{t \to s} \int C \to 0.
\]

Hence $\mathcal{F}$ is cts on $C \setminus \sigma(a)$.

Since $\mathcal{F}(\lambda) = \mathbb{L}(\alpha)$ a.e. on $C \setminus \sigma(a)$, they must agree everywhere on $C \setminus \sigma(a)$. \[\square\]
Theorem: Suppose $a \in M$. Then:

(i) For any polynomial $p$,

$$M_p(a) = Ma \# P$$

$$\left( M_p(a)(A) \leq Ma\left( P^{\ast}(A) \right) \right)$$

(ii) $Ma^* = Ma \#^{-1}$

(iii) If $a$ is invertible

$$Ma^{-1} = Ma \#^{-1}$$

Proof: Suppose $p$ is a (non-constant) polynomial and $\lambda \in \mathbb{C}$.

Take $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ so that

$$p(a) - \lambda = a (a - \lambda_1) (a - \lambda_2) \cdots (a - \lambda_n)$$

Then
\[
\int_C \log |z - \lambda| \, d\mu_{\pi}(z) = \int \left( p(a) - \lambda \right)
\]

\[
= \int \left( \alpha (a - \lambda_1) \cdots (a - \lambda_n) \right)
\]

\[
= \int \left( \alpha + \int (a - \lambda_1) + \cdots + \int (a - \lambda_n) \right)
\]

\[
= \log |\alpha| + \sum_{j=1}^{n} \int_C \log |z - \lambda_j| \, d\mu_a(z)
\]

\[
= \int_C \log |\alpha (z - \lambda_1) \cdots (z - \lambda_n)| \, d\mu_a(z)
\]

\[
= \int_C \log |p(z) - \lambda| \, d\mu_a(z)
\]

\[
= \int_C \log |\gamma - \lambda| \, d(\mu_a \# p)(\gamma).
\]

Hence \( \mu_{\pi}(a) = \mu_a \# p \).