The Circular Operator: 28 May 2021

$G_N = (g_{ij})$ g_{ij} i.i.d. random variables

$E(g_{ij}) = 0$ $E(|g_{ij}|^2) = 1$. $X_N = \frac{1}{\sqrt{N}} G$

eigenvalue distribution of X_N converges to uniform measure on $\mathbb{D}$. If Y_N is another matrix like X_N then $X_N Y_N^{-1}$ converges to Cauchy dist.

M = finite von Neumann algebra

φ = faithful normal trace, $\varphi(1) = 1$

$S_1, S_2 \in M$ free and semi-circular $\varphi(S_i^{2n}) = \int_{-2}^{2} t^{2n} \frac{\sqrt{4-t^2}}{2\pi} dt$

$\varphi(S_i^{2n-1}) = 0$.

$C = \dfrac{S_1 + i S_2}{\sqrt{2}}$, μ_C = uniform measure on $\mathbb{D}$.

$C^*C \in M$, find spectral measure

$$\varphi((C^*C)^n) = \cdots = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

$$G(z) = \text{Cauchy trans.} = \sum_{n \geq 0} \frac{\varphi((C^*C)^n)}{z^{n+1}}$$

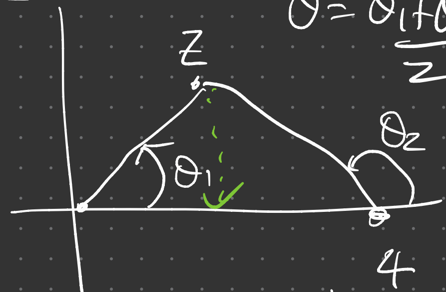
$|z| \geq \|C\|^2$ $= \dfrac{z - \sqrt{z^2 - 4z}}{2z}$. For $\varepsilon > 0$

$$-\frac{1}{\pi}\mathrm{Im}(G(x+i\varepsilon))$$

$$= \frac{|(x+i\varepsilon)^2 - 4(x+i\varepsilon)|^{1/2}}{2\pi(x^2+\varepsilon^2)}\{x\sin\theta - \varepsilon\cos\theta\}$$

$$\to \begin{cases} \dfrac{\sqrt{4x-x^2}}{2\pi x} & 0 \leq x \leq 4 \\ 0 & \text{other.} \end{cases}$$

$\theta = \dfrac{\theta_1+\theta_2}{2}$

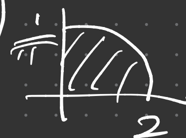


$$\sqrt{z^2-4z} = |z^2-4z|^{1/2} e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}$$

[Steiltjes inversion]

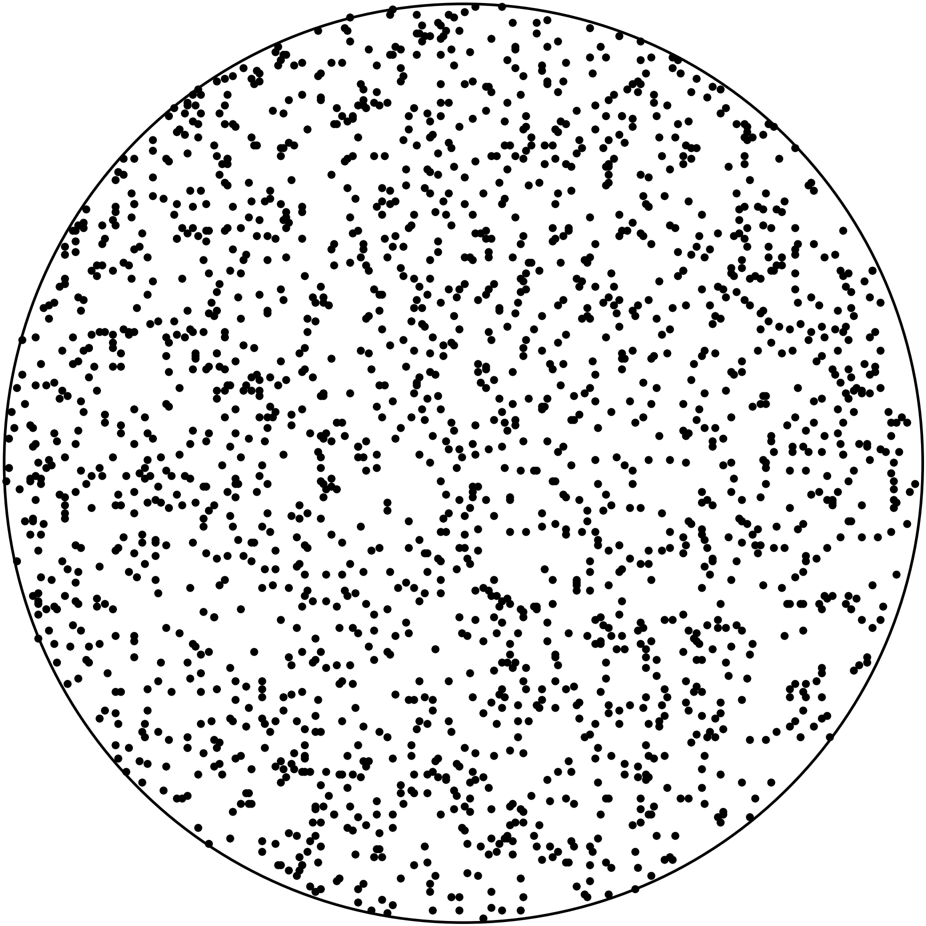
$$\int_0^4 x^n \frac{\sqrt{4x-x^2}}{2\pi x}\,dx = \frac{1}{n+1}\binom{2n}{n}.$$

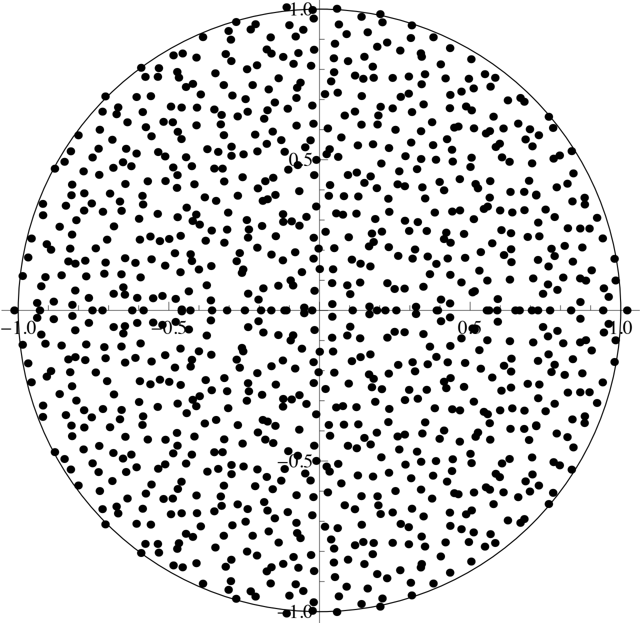
As we saw $\displaystyle\int f(\sqrt{t})\,d\mu_{|c|^2}(t)$



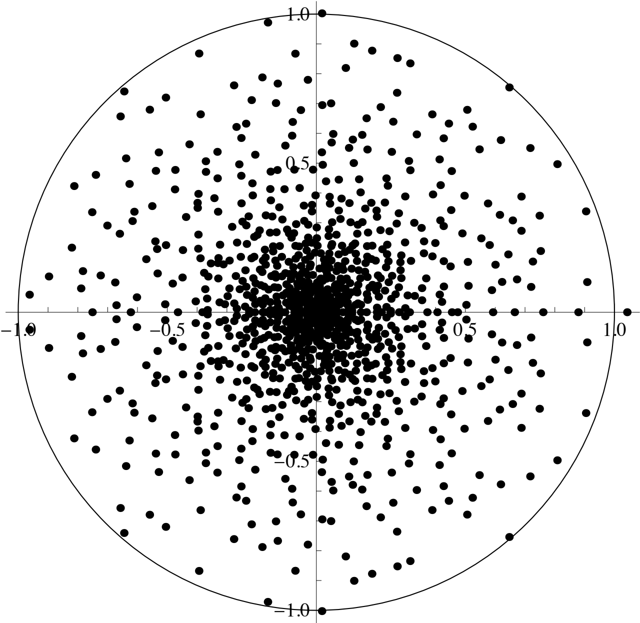
$$= \int f(t)\,d\mu_{|c|}(t) \Rightarrow$$

$$\varphi(|c|^n) = \int_0^2 \frac{\sqrt{4-t^2}}{\pi} t^n\,dt$$





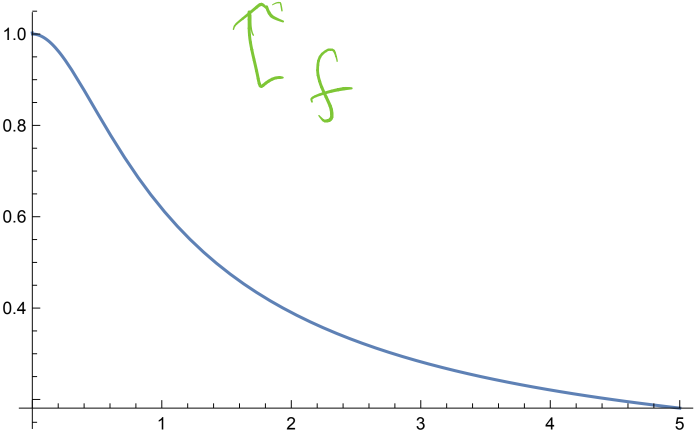
1.0
0.5
−0.5
−1.0
−1.0
−0.5
0.5
1.0



1.0
0.5
−0.5
−1.0
−1.0
−0.5
0.5
1.0

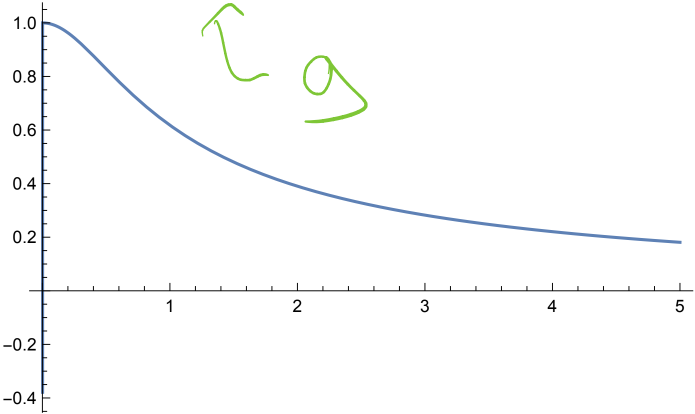
In[•]:= $\text{Plot}\left[\frac{-1+\sqrt{1+4\,v^2}}{2\,v^2},\ \{v,\ 0,\ 5\}\right]$

f

Out[•]=


In[•]:= $\text{Plot}\left[\frac{1-f[v]}{v^2\,f[v]},\ \{v,\ 0,\ 5\}\right]$

g

Out[•]=


R-diagonal operator

$c = u|c|$ polar decomposition

u & $|c|$ are $*$-free

u is a Haar unitary

$$\varphi(u^n) = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases} = \int_{\mathbb{T}} z^n \, dz$$

Another Picture of a Circular Operator

$b =$ Bernoulli random variable $\in M$

$b = b^*$, $b^2 = 1$, $\varphi(b) = 0$, $\varphi(b^2) = 1$

$x =$ semi-circular operator with radius 2

x & b are free $\quad a = bx = b \cdot \mathrm{sgn}(x)|x|$

Claim: a is a circular operator

$a^* a = (bx)^*(bx) = x^2$ ④

$|a| = |x|$

$\sigma(c) = \sigma(e^{i\theta} c) \quad \forall\, \theta \in \mathbb{R}$

$\sigma(c)$ is notationally invariant

$$L(c-\lambda) = L(c - \omega|\lambda|)$$
$$= L(\omega(\bar{\omega}c - |\lambda|))$$
$$= L(\omega) + L(\bar{\omega}c - |\lambda|)$$
$$= L(c - |\lambda|)$$

So μ_c is rotationally invariant.

Result $L(c-\lambda) = \frac{1}{2}(\lambda\bar{\lambda} - 1)$ for $|\lambda| < 1$ and $\log|\lambda|$ for $|\lambda| > 1$.

$$\frac{1}{2\pi} \nabla^2 L(c-\lambda) = \frac{1}{\pi} \frac{\partial^2}{\partial\bar{\lambda}\partial\lambda}(\lambda\bar{\lambda} - 1) = \frac{1}{\pi}$$

Suppose c_1 & c_2 are free and circular $a = c_1 c_2^{-1}$ 5

$$d\mu_a(\lambda) = \frac{1}{\pi} \frac{1}{(1+\lambda\bar{\lambda})^2} d\lambda$$

$$d\mu_{|a|}(\lambda) = \frac{2}{\pi} \frac{1}{1+\lambda^2} d\lambda \text{ on } [0,\infty)$$

Key Ideas of the Proof

$b = b^*$, $b^2 = 1$ Bernoulli

$x =$ semi-circle, b & x are free

$c = bx$. $L(c-\lambda) = L(c+\lambda)$

$= L(bx+\lambda) = L(b(x+\lambda b))$

$= L(b) + L(x+\lambda b) = L(x+\lambda b)$

We suppose $\lambda > 0$.

Let $M_x(s) = \varphi((1-sx)^{-1}) \quad s \in \mathbb{C}$ (6)

$$= \sum_{n \geq 0} \varphi(x^n) s^n$$

$x(s) = (1-sx)^{-1} - M_x(s)$, $\varphi(x(s)) = 0$.

$$M_{\lambda b}(t) = \varphi((1-t\lambda b)^{-1}) = \sum_{n \geq 0} \varphi(\lambda^n b^n) t^n$$

$$b(t) = (1-t\lambda b)^{-1} - M_{\lambda b}(t)$$

$$\varphi(b(t)) = 0$$

$\rho^{-1} = M_x(s) M_{\lambda b}(t)$. Fact: If

$s M_x(s) = t M_{\lambda b}(t)$ and

$M_x(s) + M_{\lambda b}(t) = 1$ then

$$\boxed{\frac{-s}{M_{\lambda b}(t)}(x+\lambda b) = (1-sx)(1-\rho\, x(s) b(t))(1-t\lambda b)}$$

$$L(x+\lambda b) = \frac{1}{2}\int \log(1+v^2u^2)\, d\mu_x(u)$$
$$+ \frac{1}{2} \log(1+\lambda^{-2}v^{-2})$$
$$+ \log(1-f(v)) - \log(v)$$

$f(v) = M_x(iv)$ v is such that

$$\lambda^2 = g(v) = \frac{1-f(v)}{v f(v)}$$

$t = \frac{i}{\lambda^2 v}$ ✓

$s = i$ ✓

$x = \begin{pmatrix} 0 & c \\ c^* & 0 \end{pmatrix}$ even

$b = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$ Bernoulli

x & b free