Notation

$$a_\lambda = a - \lambda I = a - \lambda$$

$$L : \mathcal{M} \longrightarrow \mathbb{R} \cup \{-\infty\}$$

$$a \longmapsto \tau(\log|a|)$$

$$L_\varepsilon : a \longmapsto \frac{1}{2} L(a^* a + \varepsilon)$$

$$\lambda \longmapsto L_\varepsilon(a_\lambda) \quad \text{is} \quad C^\infty \text{ in } \lambda,$$

and

$$\nabla^2 L_\varepsilon(a_\lambda) = 4 \frac{d^2}{d\bar{\lambda} d\lambda} L_\varepsilon(a_\lambda)$$

$$= \frac{1}{2} \varepsilon \, \tau\left((a_\lambda^* a_\lambda + \varepsilon)^{-1} (a_\lambda a_\lambda^* + \varepsilon)^{-1} \right)$$

$$> 0.$$

Hence $L_\varepsilon(a_\lambda)$ is subharmonic in λ.

$L_\varepsilon \searrow L$ as $\varepsilon \to 0$.

Hence L too is subharmonic.

Def^n: For $a \in \mathcal{M}$, and $\varepsilon > 0$, we define

$$d\mu_{a,\varepsilon} = \frac{1}{2\pi} \nabla^2 L_\varepsilon(a_\lambda)\, d\lambda$$

$$d\mu_a = \frac{1}{2\pi} \nabla^2 L(a_\lambda)\, d\lambda.$$

So give $\varphi \in C_c^\infty(\mathbb{C})$

$$\langle d\mu_{a,\varepsilon}, \varphi \rangle = \frac{1}{2\pi}\int_{\mathbb{C}} \nabla^2 \varphi(\lambda)\, L_\varepsilon(a_\lambda)\, d\lambda$$

$$\langle d\mu_a, \varphi \rangle = \frac{1}{2\pi}\int_{\mathbb{C}} \nabla^2 \varphi(\lambda)\, L(a_\lambda)\, d\lambda.$$

The measure $d\mu_a$ is called the Brown measure of a.

Lemma: If $d\mu$ is a measure given by $\langle d\mu, \varphi \rangle = \int \nabla^2 \varphi(\lambda) f(\lambda)\, d\lambda$ and if f is C^2 with $\nabla^2 f \geq 0$ on an open set B, then for $A \subseteq B$ Borel,

$$\mu(A) = \int_A \nabla^2 f(\lambda)\, d\lambda.$$

Pf: If $A \subseteq B$ is open, find $\varphi_n \nearrow 1_A$ in $C_c^\infty(B)^+$. Then

$$\mu(A) = \int 1_A \, d\mu = \int \lim_n \varphi_n \, d\mu \overset{(MCT)}{=} \lim_n \int \varphi_n \, d\mu$$

$$= \lim_n \int \nabla^2 \varphi_n(\lambda) \, f(\lambda) d\lambda$$

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$$= \int_A \nabla^2 f(\lambda) \, d\lambda.$$

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Lemma:

$\lambda \mapsto L(a_\lambda)$ is harmonic outside of $\sigma(a)$.

Pf: If $\lambda_0 \in \mathbb{C} \setminus \sigma(a)$, then for $\lambda \approx \lambda_0$

$$L(a_\lambda) = L(a_{\lambda_0} a_{\lambda_0}^{-1} a_\lambda)$$

$$a_\lambda = a - \lambda$$
$$= a - \lambda_0 + \lambda_0 - \lambda$$
$$= a_{\lambda_0} + \lambda_0 - \lambda$$

$$= L(a_{\lambda_0}) + L(a_{\lambda_0}^{-1} a_\lambda)$$

$$= L(a_{\lambda_0}) + L(a_{\lambda_0} + \lambda_0 - \lambda)$$

$$= 2L(a_{\lambda_0}) + L(1 + (\lambda_0 - \lambda) a_{\lambda_0}^{-1})$$

$$= 2L(a_{\lambda_0}) + \tau(\log|1 + (\lambda_0 - \lambda) a_{\lambda_0}^{-1}|)$$

For $(\lambda - \lambda_0)$ small,

$$\tau(\log|1 + (\lambda_0 - \lambda) a_{\lambda_0}^{-1}|) = \operatorname{Re} \tau(\log(1 + (\lambda_0 - \lambda) a_{\lambda_0}^{-1}))$$

is the real part of a

holomorphic function, and so harmonic. $\boxtimes$

Hence $\nabla^2 L(a_\lambda)$ vanishes on $\mathbb{C} \setminus \sigma(a)$

and so $\mu(\mathbb{C} \setminus \sigma(a)) = \int_{\mathbb{C} \setminus \sigma(a)} \nabla^2 L(a_\lambda) d\lambda = 0.$

Prop: $d\mu_{a,\varepsilon} \xrightarrow{w^*} d\mu_a$.

I.e., for any $\varphi \in C_c^\infty$ we have

$$\langle d\mu_{a,\varepsilon}, \varphi \rangle \xrightarrow{\varepsilon \searrow 0} \langle d\mu_a, \varphi \rangle.$$

Pf:

$$\langle d\mu_{a,\varepsilon}, \varphi \rangle = \frac{1}{2\pi} \int \varphi(\lambda) \nabla^2 L_\varepsilon(a_\lambda) d\lambda$$

$$= \frac{1}{2\pi} \int \nabla^2 \varphi(\lambda) L_\varepsilon(a_\lambda) d\lambda$$

$$\xrightarrow[\text{MCT}]{\varepsilon \searrow 0} \frac{1}{2\pi} \int \nabla^2 \varphi(\lambda) L(a_\lambda) d\lambda = \langle d\mu_a, \varphi \rangle.$$

Theorem: $d\mu_{a,\varepsilon}$ is a probability, i.e.,

$$\mu_{a,\varepsilon}(\mathbb{C}) = 1.$$

Pf: Set $g_\varepsilon(\lambda) = 2\frac{d}{d\lambda}L_\varepsilon(a_\lambda)$

$$= -\tau\left((a_\lambda^* a_\lambda + \varepsilon)^{-1} a_\lambda^*\right).$$

For p a polynomial

$$\frac{d}{d\lambda}\tau\left(p(a_\lambda^* a_\lambda)\right) = -\tau\left(p'(a_\lambda^* a_\lambda)\, a_\lambda^*\right).$$

Since we have a power series representation valid on a neighbourhood of λ and ε (with $\mathrm{Re}(\varepsilon) > 0$) we get this identity for $L_\varepsilon(a_\lambda) = \tau\left(\log|a_\lambda^* a_\lambda + \varepsilon|\right)$ too.

For $|\lambda| > \|a\|$, set $g(\lambda) = -\tau\left((a-\lambda)^{-1}\right)$

$$= \lambda^{-1}\left(\sum_{j=0}^{\infty}\frac{\tau(a^j)}{\lambda^j}\right).$$