Notes on Brown Measure

1 Brown Measure for Matrices

Let $T \in M_N(\mathbb{C})$ have eigenvalues $\lambda_1, \dots, \lambda_N$ repeated according to multiplicity. Also for any matrix let $|T| = \sqrt{T^*T}$. Recall that $g(x,y) = \log\sqrt{x^2+y^2}$ is locally integrable, even on a neighbourhood of $(0,0)$. Let $C_c^\infty(\mathbb{R}^2)$ be the space of C^∞ functions with compact support. $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

Theorem

$$\int \nabla^2 f(x,y) \log[\det|x+iy-T|]\,dx\,dy$$

$$= 2\pi \sum_{i=1}^{N} f(\lambda_i) \qquad (*)$$

Notation Let $\mu_T = \dfrac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$. Then $(*)$ says that $\mu_T = \dfrac{1}{2\pi N}\nabla^2 \log[\det|\lambda - T|]$

where the RHS is interpreted in the distributional sense. Let $\mathrm{tr} = \frac{1}{N}\mathrm{Tr}$. Then

$$\boxed{\mu_T = \frac{1}{2\pi}\nabla^2 \mathrm{tr}(\log|\lambda - T|)}$$

(2)

§2 Extensions to Finite von Neumann algebras

H = separable Hilbert space

$B(H)$ = bounded linear operators on H

$T_n \xrightarrow{WOT} T$ means $\forall\, \xi, \eta \in H$ $\langle (T_n - T)\xi, \eta \rangle \to 0$

WOT = weak operator topology (metrisable)

$M \subseteq B(H)$ self-adjoint WOT closed subalgebra

is a von Neumann algebra. Let $\tau : M \to \mathbb{C}$

be a linear functional such that

- $\tau(ab) = \tau(ba)$ (tracial)
- $\tau(x^*x) \geq 0$ (positive)
- $\tau(x^*x) = 0 \Rightarrow x = 0$ (faithful)
- $\tau(1) = 1$ (normalized)
- $a_n \xrightarrow{WOT} a \Rightarrow \tau(a_n) \to \tau(a)$ (normal);

τ is a faithful normal trace. (M, τ) is a non-commutative probability space.

$M_2 \ni x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in M_2 \otimes M_2$ $\tau(x) = \tau \otimes \tau \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$

so τ extends to a trace on $\bigotimes_{i=1}^{\infty} M_2$. We can complete to form $M = \overline{\bigotimes_1^{\infty} M_2}^{WOT}$ and τ a faithful normal trace on M.

§3 Brown Measure

If $a = a^* \in M$ then $\sigma(a) \subseteq \mathbb{R}$ and $\exists$ μ_a a probability measure on $\sigma(a)$ such that $\tau(f(a)) = \int f(t)\, d\mu_a(t)$ for all ess. bounded Borel functions f. μ_a is the spectral measure of a with respect to τ.

If $a \in M$ let $\mu_{|a|}$ be the spectral measure of $|a| = \sqrt{a^* a}$. Let

$$L(a) = \int \log(t)\, d\mu_{|a|}(t) \in [-\infty, \|a\|]$$

$$= \tau(\log|a|) \quad \text{(as an unbounded operator)}$$

$$\mu_a = \frac{1}{2\pi} \nabla^2 \tau(\log|x+iy-a|)$$

in the sense of distributions.

Thm: μ_a is a compactly supported regular Borel measure on $\mathbb{C}$ with support in $\sigma(a)$. We have

- $\tau(a^n) = \int t^n\, d\mu_a(t)$
- $L(a-\lambda) = \int \log|t-\lambda|\, d\mu_a(t)$ $\Big\{$ uniquely determines μ_a $\Big\}$

④

§4 Examples

s_1 & s_2 free & semi-circular $c = \frac{1}{\sqrt{2}}(x_1 + i x_2)$

is circular μ_c = uniform probability measure

on $\overline{\mathbb{D}} = \{z \mid |z| \leq 1\}$.

$u_1 \ldots u_n$ free Haar unitaries

$s = u_1 + \cdots + u_n$. μ_s has support

$\{z \mid |z| \leq \sqrt{n}\}$, is rotationally invariant

and has radial density

$$f(r) = \begin{cases} 0 & r > \sqrt{n} \\ \dfrac{n^2(n-1)}{\pi(n^2 - r^2)^2} & r \leq \sqrt{n} \end{cases}$$

Proof of Theorem 1

Let $f \in C_c^\infty(\mathbb{R})$ we claim:

$$\int \nabla^2 f(x,y) \log\sqrt{x^2+y^2}\, dx\,dy = 2\pi f(0,0).$$

- $D_R = \{(x,y) \mid x^2+y^2 < R^2\}$ open disc

  $D_{r,R} = \{(x,y) \mid r^2 < x^2+y^2 < R^2\}$ open annulus

Calculus lemma: $f\nabla^2 g - g\nabla^2 f = \nabla\cdot(f\nabla g - g\nabla f)$.

for smooth enough f & g. If $g(x,y) = \log\sqrt{x^2+y^2}$

$\nabla^2 g = 0 \quad \nabla g = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \Big| \; \vec{n} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$ on ∂D_r

$\nabla g\cdot\vec{n} = \frac{1}{r}$

Choose R large enough so that $\mathrm{supp}(f) \subseteq D_R$

$$\int_{D_{r,R}} \nabla^2 f\, g\, dx\,dy = \int_{D_{r,R}} \nabla\cdot(g\nabla f - f\nabla g)\, dx\,dy$$

$$= \int_{\partial D_{r,R}} (g\nabla f - f\nabla g)\cdot\vec{n}\, ds = \int_{\partial D_{r,R}} g\nabla f\cdot\vec{n} - f\nabla g\cdot\vec{n}\, ds$$

$$= \int_{\partial D_R} g\nabla f\cdot\vec{n} - f\nabla g\cdot\vec{n}\, ds - \int_{\partial D_r} g\nabla f\cdot\vec{n} - f\nabla g\cdot\vec{n}\, ds$$

$$= \frac{1}{r}\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, r\,d\theta - r\log r\int_0^{2\pi} \frac{\partial f}{\partial r}(r\cos\theta, r\cos\theta)\, d\theta$$

$$\to \frac{1}{r}\, 2\pi r\, f(0,0) - 0 = 2\pi f(0,0) \quad \text{as } r\to 0$$

thus $\int(\nabla^2 f)\, g\, dx\,dy = \lim_{\substack{r\to 0\\ R\to\infty}} \int_{D_{r,R}} \nabla^2 f\, g\, dx\,dy = 2\pi f(0,0)$

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$$\int \nabla^2 f(x,y) \log\sqrt{(x-x_0)^2+(y-y_0)^2}\, dx\,dy$$

$$= \int \nabla^2(f)(x+x_0, y+y_0) \log\sqrt{x^2+y^2}\, dx\,dy = 2\pi f(x_0,y_0).$$

$$\det(\lambda - T) = \prod_{i=1}^{N} (\lambda - \lambda_i) \qquad \{\lambda_i\}_{i=1}^{N} \text{ eigenvalues of } T$$

$$\det|\lambda - T| = |\det(\lambda - T)| = \prod_{i=1}^{N} |\lambda - \lambda_i|$$

$$\log(\det|\lambda - T|) = \sum_{i=1}^{N} \log|\lambda - \lambda_i|$$

$$\mathrm{tr}(\log|\lambda - T|) = \frac{1}{N}\mathrm{Tr}(\log|\lambda - T|) = \frac{1}{N}\log(\det|\lambda - T|)$$

$$= \frac{1}{N}\sum_{i=1}^{N} \log|\lambda - \lambda_i|.$$

$$\frac{1}{2\pi}\mathrm{tr}(\log|\lambda - T|) = \frac{1}{2\pi N}\sum_{i=1}^{N} \log|(\lambda - \lambda_i)| \qquad \{\lambda = x+iy$$

$$\int \nabla^2 f(x,y) \cdot \left[\frac{1}{2\pi}\mathrm{tr}(\log|\lambda - T|)\right] dx\,dy$$

$$= \frac{1}{2\pi N}\sum_{i=1}^{N} \int \nabla^2 f(x,y) \log|\lambda - \lambda_i|\, dx\,dy$$

$$= \frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) = \frac{1}{N}\sum_{i=1}^{N} \int f(x,y)\, \delta_{\lambda_i}(x,y)\, dx\,dy$$

$$= \int f(z)\, d\mu_T(z) \qquad \text{where}$$

$$\mu_T = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i} = \text{empirical measure of } T.$$

⑥ **Corollary** $\mu_T = \frac{1}{2\pi} \nabla^2 \operatorname{tr}(\log|\lambda - T|)$

Construction of Brown Measure

(M, τ) is a finite von Neumann algebra with faithful normal trace τ. For $a = a^*$

μ_a is the spectral measure of a:

μ_a satisfies $\tau(f(a)) = \int f(t)\, d\mu_a(t)$

for all bounded Borel functions on $\sigma(a)$

$\subseteq [-\|a\|, \|a\|]$. For $a \in M$,

$|a| = \sqrt{a^* a}$. Let $L(a) = \int_{[1, \|a\|]} \log(t)\, d\mu_{|a|}(t)$

$+ \int_{[0,1]} \log(t)\, d\mu_{|a|}(t)$ where the second

integral is interpreted as an extended real number. We write this as

$$L(a) = \int \log(t)\, d\mu_{|a|}(t).$$

Lemma

(i) $\mu_{|a|} = \mu_{|a^*|}$

(ii) $L(a) = L(a^*)$

(iii) $L(a^*a) = 2L(a)$

(iv) for $0 \le a \le b$, $L(a) \le L(b)$

(v) $L(a) = \inf_{\varepsilon > 0} L_\varepsilon(a) = \inf_{\varepsilon > 0} \frac{1}{2} L(a^*a + \varepsilon)$ and $L : M \to [-\infty, \infty)$ is upper semi-continuous.

Proof: Suppose $M \subseteq B(H)$ define v by

(i) $v(|a|\xi) = a\xi$. As $\| |a| \xi \| = \| a\xi \|$, v extends to a partial isometry: $\ker(a)^\perp \to \overline{\operatorname{ran}(a)}$, and $a = v|a|$. Then $a^* = |a| v^* = v^*(v|a|v^*)$. By the uniqueness of the polar decomposition $v|a|v^* = |a^*|$.

Let $p = \operatorname{Proj}(\ker(a)^\perp)$ & $q = \operatorname{Proj}(\overline{\operatorname{ran}(a)})$ then $v^*v = p$ & $vv^* = q$. Since M is assumed to be finite $\exists\, w$ s.t. $w^*w = 1-p$ and $ww^* = 1-q$.

Let $u = v + w$ then $|a^*| = u|a|u^*$ so $\tau(|a^*|^n) = \tau(u|a|^n u^*) = \tau(|a|^n)$. Hence $\mu_{|a|} = \mu_{|a^*|}$.

(ii) $L(a^*) = \int \log(t)\, d\mu_{|a^*|}(t) = \int \log(t)\, d\mu_{|a|}(t) = L(a)$.

(iii) Suppose $x \in M$ and $x \ge 0$.

⑧ Let us show that $\int f(t)\,d\mu_x(t) = \int f(\sqrt{t})\,d\mu_{x^2}(t)$.
It is enough to prove this for $f = \mathbb{1}_A$, $A \subseteq \mathbb{R}$ Borel. Let $A^2 = \{t^2 \mid t \in A\}$ & $g(t) = t^2$. Then $\mathbb{1}_{A^2} \circ g = \mathbb{1}_A$. Then $\int \mathbb{1}_A(\sqrt{t})\,d\mu_{x^2}(t)$

$$= \int \mathbb{1}_{A^2}(t)\,d\mu_{x^2}(t) = \tau(\mathbb{1}_{A^2}(x^2))$$

$$= \tau(\mathbb{1}_{A^2} \circ g(x)) = \tau(\mathbb{1}_A(x)) = \int \mathbb{1}_A(t)\,d\mu_x(t)$$

If $\log \notin L^1(\mu_{|a|})$ then $L(a), L(a^*a) = -\infty$.
If $\log \in L^1(\mu_{|a|})$ then $L(a) = \int \log(t)\,d\mu_{|a|}(t)$

$$= \int \log(\sqrt{t})\,d\mu_{a^*a}(t) = \frac{1}{2}\int \log(t)\,d\mu_{a^*a}(t)$$

$$= \frac{1}{2} L(a^*a).$$

(iv) A lemma in functional calculus: $A = (t, \infty)$
$e = \mathbb{1}_A(a)$ $f = \mathbb{1}_A(b)$. Then it can be shown there is a partial isometry v s.t. $v^*v = e$ & $vv^* \leq f$.

Thus for $t > 0$ $\mu_a([0,t]) = 1 - \mu_a(A)$

$$= 1 - \tau(e) \geq 1 - \tau(f) = 1 - \mu_b(A) = \mu_b([0,t]).$$

Let s be a simple function of the form

$s(t) = \alpha + \sum_1^n \alpha_i \mathbb{1}_{A_i}$. $\alpha \in \mathbb{R}$, $\alpha_i \leq 0$. Then

$\int s(t)\,d\mu_a(t) \leq \int s(t)\,d\mu_b(t) \Rightarrow L(a) \leq L(b)$.

(v) Fix $\varepsilon > 0$. $a \mapsto a^*a + \varepsilon$ is norm continuous $M \to M$, $\log : M_+^{-1} \to \mathbb{R}$ is norm continuous. Thus $L_\varepsilon : M \to \mathbb{R}$ is norm continuous. As log is increasing

$$2L_\varepsilon(a) = \tau(\log(a^*a+\varepsilon)) = \int \log(t+\varepsilon)\, d\mu_{a^*a}(t)$$
$$\xrightarrow{\text{decreases}} \int \log(t)\, d\mu_{a^*a}(t) = L(a^*a) = 2L(a).$$

Thus $\lim_{\varepsilon \to 0^+} L_\varepsilon(a) = L(a)$. By (iv) $L_{\varepsilon'}(a) \leq L_\varepsilon(a)$ for $\varepsilon' \leq \varepsilon$. Hence

$L(a) = \inf_{\varepsilon > 0} L_\varepsilon(a)$. Now L is the inf of family of continuous functions, hence L is upper semi-continuous.(†)

Corollary

(i) if $d\mu_{|a|}(t) = f(t)\,dt$ for $f \in \bigcup_{1 < p \leq \infty} L^p(\mathbb{R})$ then $L(a) > -\infty$.

(ii) If $\exists \{x_n\}_{n=1}^{\infty} \subseteq [0,\infty)$ x_n decreasing to 0 with $\sum_{n \geq 1} \mu_{|a|}(\{x_n\}) \log(x_n) = -\infty$, then $L(a) = -\infty$.

(†) $f^{-1}((-\infty, t))$ open $\forall t \iff f$ is u.s.c. $\iff \left\{ \begin{array}{l} f(t) < \alpha \Rightarrow f(s) < \alpha \\ \text{for } s \text{ close to } t \end{array} \right\}$

(10) (iii) If $\mu_{|a|}(\{0\}) > 0$ then $L(a) = -\infty$

(iv) if $z \in \mathbb{C} \setminus \{0\}$ $L(z1) = \log|z|$

(v) $L(0) = -\infty$

(vi) if a is unitary $L(a) = 0$

(vii) if $a \in M^{-1}$, $L(a) > -\infty$

Proof: (i) Suppose $1 < p < \infty$ $f \in L^p(\mathbb{R})$ and $d\mu_{|a|}(t) = f(t)\,dt$. Then $|L(a)|$

$$\leq \int_{\sigma(|a|)} |\log(t)|\, d\mu_{|a|}(t) = \int_{\sigma(|a|)} |\log(t)|\, f(t)\, dt$$

$$\leq \left(\int_{\sigma(|a|)} |\log(t)|^q dt \right)^{1/q} \left(\int_{\sigma(|a|)} |f(t)|^p dt \right)^{1/p} < \infty.$$

If $p = \infty$ then $\int_{\sigma(|a|)} |\log(t)|\, f(t)\, dt$

$$\leq \|f\|_\infty \int_{\sigma(|a|)} |\log(t)|\, dt < \infty.$$

(ii) Let $X = \{x_n \mid n \geq 1\} \subseteq [0,1]$ WLOG.

$$\infty = \sum_{n \geq 1} -\log(x_n)\, \mu_{|a|}(\{x_n\}) = \int_X -\log(t)\, d\mu_{|a|}(t)$$

$$\leq \int_{[0,1]} -\log(t)\, d\mu_{|a|}(t).$$ This implies

that $\int_{[0,1]} \log(t)\, d\mu_{|a|}(t) = -\infty$ and

thus $L(a) = -\infty$.

(iii) If $\mu_{|a|}(\{0\})>0$ then

$$\infty = \log(0)\,\mu_{|a|}(\{0\}) = \int_{\{0\}} -\log(t)\,d\mu_{|a|}(t)$$
$$\leq \int_{[0,1]} -\log(t)\,d\mu_{|a|}(t).$$

(iv) Suppose $z \in \mathbb{C}\setminus\{0\}$ and $a = z1$.
$|a| = |z|1$ and $\mu_{|a|} = \delta_{|z|}$.
$L(a) = \int \log(t)\,d\mu_{|a|}(t) = \log(|z|)$.

(v) $L(0) = -\infty$ by (iii)

(vi) If a is a unitary $a^*a=1$ and $|a|=1$
Thus $\mu_{|a|} = \delta_1$ so $L(a) = \int \log(t)\,d\mu_{|a|}(t)$
$= \log(1) = 0$.

(vii) Suppose $a \in M^{-1}$ then $\exists\, m>0$ s.t.
$\sigma(|a|) \subseteq [m, \|a\|]$ and so $\int \log(t)\,d\mu_{|a|}(t)$
$= \int_{[m,\|a\|]} \log(t)\,d\mu_{|a|}(t)$ is finite.

Definition Let $I \subseteq \mathbb{R}$ be an open interval and $a: I \to M$ a function. We say that a is a **C^1-function** if $\exists$ $b: I \to M$ continuous such that $\forall\, t_0 \in I$
$$a(t) - a(t_0) = (t-t_0)\,b(t_0) + O((t-t_0)^2)$$

(12) **Lemma** Suppose $f \in C^1(\mathbb{R})$ and $a: I \to M$ is C^1. Then $I \ni t \mapsto \tau(f(a(t)))$ is C^1 and
$$\frac{d}{dt}\tau(f(a(t))) = \tau(f'(a(t))\,a'(t))$$

Proof: Suppose first that $f(t) = t^n$. Then
$$a(t+h)^n = [a(t) + h\,b(t) + O(h^2)]^n$$
$$= a(t)^n + h\sum_{k=0}^{n-1} a(t)^k b(t) a(t)^{n-k-1} + O(h^2)$$
Hence
$$\frac{a(t+h)^n - a(t)^n}{h} = \sum_{k=0}^{n-1} a(t)^k b(t) a(t)^{n-k-1} + O(h)$$
$$\frac{d}{dt}\tau(f(a(t))) = n\,\tau(a(t)^{n-1}b(t)) = \tau(f'(a(t))\,a'(t))$$
So the result holds for polynomials. For general f choose a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ such that $p_n \to f$ & $p_n' \to f'$ uniformly on $[-M, M]$ where M is chosen large enough so that $\sigma(a(t+h)) \subseteq [-M, M]$ for h small. ▨

Theorem (i) If $x \in M^{-1}$ and $a \in M$ then
$$L(xe^a) = L(x) + \mathrm{Re}(\tau(a))$$
(ii) For all $a, b \in M$ $L(ab) = L(a) + L(b)$.

Proof: Recall that for a in a Banach algebra $e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$. Then $\frac{d}{dt} e^{ta} = a e^{ta} = e^{ta} a$.

Let $b(t) = \tau(\log(x e^{ta} e^{ta^*} x^*))$. Since $x e^{ta} e^{ta^*} x^* > 0$ i.e. is positive and invertible b is continuous. So $\exists\, \beta > 0$ such that for $t \in [0,1]$ $\sigma(x e^{ta} e^{ta^*} x^*) \subseteq [0, \beta]$. Also $t \mapsto (x e^{ta} e^{ta^*} x^*)^{-1}$ is norm continuous so there is $\alpha > 0$ so that $\sigma((x e^{ta} e^{ta^*} x^*)^{-1}) \subseteq [0, \alpha^{-1}]$ for $t \in [0,1]$. Hence $\forall t \in [0,1]$, $\sigma(x e^{ta} e^{ta^*} x^*) \subseteq [\alpha, \beta]$.

Now $\log$ is C^1 on $[\alpha, \beta]$ so b is C^1 on $[0,1]$. and $b'(t) = \tau\left((x e^{ta} e^{ta^*} x^*)^{-1} \left[x \frac{d}{dt}[e^{ta} e^{ta^*}] x^*\right]\right)$

$$= \tau(x^{*-1} e^{-ta^*} e^{-ta} x^{-1} x\, x^{ta} a\, e^{ta^*} x^*)$$

$$+ \tau(x^{*-1} e^{-ta^*} e^{-ta} x^{-1} x\, e^{ta} a^* e^{ta^*} x^*) = 2\mathrm{Re}(\tau(a))$$

Now $b(1) - b(0) = \int_0^1 b'(t)\, dt = 2\mathrm{Re}(\tau(a))$

$$\tau(\log(x e^{a} e^{a^*} x^*)) = b(1) = b(0) + 2\mathrm{Re}(\tau(a))$$
$$= \tau(\log(x x^*)) + 2\mathrm{Re}(\tau(a))$$

Thus $\tau(\log|x e^a|) = \tau(\log|x|) + \mathrm{Re}(\tau(a))$

or $L(x e^a) = L(x) + \mathrm{Re}(\tau(a))$

In particular $L(e^a) = \mathrm{Re}(\tau(a))$.

(14) By induction

$$L(e^{a_1}\cdots e^{a_n}) = L(e^{a_1}\cdots e^{a_{n-1}}) + \mathrm{Re}(\tau(a_n))$$

$$= \cdots = \mathrm{Re}(\tau(a_1)) + \cdots + \mathrm{Re}(\tau(a_n))$$

$$= L(e^{a_1}) + \cdots + L(e^{a_n})$$

Suppose $a, b \in M^{-1}$. We may write $a = u|a|$ with u unitary and $|a| \in M^{-1}$. Let $h_a \in M_{sa}$ be such that $u = e^{h_a}$ ($h = \log(u)$, as $\log$ is a bounded Borel function on $\mathbb{T}$). The same for b.

$$L(ab) = L(e^{h_a} e^{\log|a|} e^{h_b} e^{\log|b|})$$

$$= \mathrm{Re}(\tau(h_a) + \tau(\log|a|) + \tau(h_b) + \tau(\log|b|))$$

$$= L(e^{h_a} e^{\log|a|}) + L(e^{h_b} e^{\log|b|})$$

$$= L(a) + L(b).$$

Suppose $a \in M$ and $b \in M^{-1}$. Now we may write $a = u|a|$, with u unitary.

Let $a_\varepsilon = u\sqrt{a^*a+\varepsilon} \in M^{-1}$, for $\varepsilon > 0$.

$\varepsilon \longmapsto \sqrt{a^*a+\varepsilon}$ is norm continuous

$\varepsilon \longmapsto u\sqrt{a^*a+\varepsilon}\, b$ is norm continuous

$\varepsilon \longmapsto L(u\sqrt{a^*a+\varepsilon}\, b)$ is usc.

$$\lim_{\varepsilon\to 0} L(u\sqrt{a^*a+\varepsilon}\,b) \le L(u|a|b) = L(ab).$$

On the other hand $L(u\sqrt{a^*a+\varepsilon}\,b)$

$$= L(u) + L(\sqrt{a^*a+\varepsilon}) + L(b)$$

$$= \tfrac{1}{2} L(a^*a+\varepsilon) + L(b)$$

$$= L_\varepsilon(a) + L(b)$$

So $\quad L_\varepsilon(a) + L(b) \le L(ab)$

$$L(a) + L(b) = \inf_{\varepsilon>0} L_\varepsilon(a) + L(b) \le L(ab).$$

Thus for b invertible and a arbitrary

$$L(a) + L(b) \le L(ab).$$

Now $\quad 0 = L(1) = L(bb^{-1}) = L(b) + L(b^{-1}).$

Thus $\quad L(b^{-1}) = -L(b)$. So

$$L(a) = L(abb^{-1}) \ge L(ab) - L(b)$$

$$\Rightarrow \quad L(a) + L(b) \ge L(ab)$$

Thus $\quad L(ab) = L(a) + L(b)$ when b is invertible.

Finally suppose $a, b \in M$ are arbitrary. We again let $a_\varepsilon = u\sqrt{a^*a+\varepsilon}$. Then $a_\varepsilon b \to ab$ in norm so $L(a_\varepsilon) + L(b) = L(a_\varepsilon b)$. So

$$\inf_{\varepsilon>0} \{ L(a_\varepsilon) + L(b) \} = \inf_{\varepsilon>0} L(a_\varepsilon b) \le L(ab).$$

here we use positivity of L $\qquad$ here we use usc of L.

Hence $L(a)+L(b) \leq L(ab)$. To get the reverse inequality we say

$$L(ab) = \frac{1}{2} L(b^*a^*ab) \overset{(\dagger)}{\leq} \frac{1}{2} L(b^*(a^*a+\varepsilon)b)$$
$$= L(|a_\varepsilon| b) = L(|a_\varepsilon|) + L(b)$$
$$= \frac{1}{2} L(a^*a+\varepsilon) + L(b) = L_\varepsilon(a) + L(b).$$

Thus $L(ab) \leq \inf_{\varepsilon>0} \{ L_\varepsilon(a) + L(b) \} = L(a) + L(b)$.

Hence for all $a, b \in M$ we have
$$L(a) + L(b) = L(ab).$$

Subharmonic Functions

Harmonic Functions

$D \subseteq \mathbb{R}^2$, $f : D \to \mathbb{R}$ a C^2 function, is harmonic if $\nabla^2 f = 0$ on D. The real & imaginary parts of analytic functions are harmonic by the Cauchy-Riemann equations.

Example $D = \mathbb{C} \setminus (-\infty, 0]$ $f(z) = \log|z|$ is harmonic as it is the real part of $\mathrm{Log}(z)$. f can be extended to a harmonic function on $\mathbb{C} \setminus \{0\}$. If f is harmonic on D it has the mean value property:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})\, d\theta \quad \text{whenever}$$

$B_r(z) \subseteq D$.

Let $$M(f, x, r) = \frac{1}{2\pi}\int_0^{2\pi} f(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(x + re^{i\theta})\, d\theta.$$

Then $$2\pi r \frac{d}{dr} M(f, x, r) = \frac{r\,d}{dr}\int_0^{2\pi} f(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta$$
$$= r\int_0^{2\pi} \frac{\partial f}{\partial r}(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta$$
$$= \int_{\partial B_r(x)} \frac{\partial f}{\partial r}\, ds = \int_{\partial B_r(s)} \nabla f \cdot \vec{n}\, ds$$
$$= \int_{B_r(x)} \nabla^2 f(y)\, dy$$

$$\boxed{2\pi r \frac{d}{dr} M(f,x,r) = \int_{B_r(x)} \nabla^2 f(y)\, dy}$$

[true for all C^2-functions]

Note that if $\nabla^2 f \equiv 0$ then

$$\frac{d}{dr} M(f,x,r) \text{ is constant}$$

and $\quad M(f,x,0) = f(x) \quad$ so

$$f(x) = M(f,x,r).$$

Subharmonic Functions

Definition $D \subseteq \mathbb{R}^2$ open, $f : D \to [-\infty, \infty)$ is *subharmonic* if

(i) f is upper semi-continuous

(ii) $\forall x \in D$ and $r > 0$ s.t. $\overline{B_r(x)} \subseteq D$ we have $f(x) \leq M(f,x,r)$ (*)

(iii) $\exists x \in D$ s.t. $f(x) > -\infty$

Remark By the monotone convergence theorem a decreasing sequence of subharmonic functions is subharmonic.

(*) By upper semi-continuity the integral $M(f,x,r)$ always converges to a number in $[-\infty, \infty)$.

Remark Suppose f is subharmonic on some open set $D \subseteq \mathbb{R}^2$ and $x \in D$. $\forall \delta > 0$ $\exists \rho > 0$ s.t. $B_\rho(x) \subseteq D$ and $f(y) \leq f(x) + \delta$ for $y \in B_\rho(x)$. Thus for $0 \leq r < \rho$

$$2\pi M(f, x, r) = \int_0^{2\pi} f(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta$$
$$\leq 2\pi [f(x) + \delta].$$

Thus for all $0 \leq r < \rho$, $M(f, x, r) \leq f(x) + \delta$

Thus $\overline{\lim\limits_{r \to 0^+}} M(f, x, r) \leq f(x)$. By (SMP)

$$f(x) \leq \underline{\lim\limits_{r \to 0^+}} M(f, x, r).$$

Hence if f is subharmonic we have

$$\lim_{r \to 0^+} M(f, x, r) = f(x).$$

So subharmonicity can be viewed as a kind of convexity in 2 dimensions.

Thm (2nd deriv. test in 2-dim)

Suppose f is C^2 on $D \subseteq \mathbb{R}^2$ (open subset)

Then f is subharmonic $\iff \nabla^2 f \geq 0$ on D

in particular $\dfrac{\nabla^2 f(x)}{4} = \lim\limits_{r \to 0^+} \dfrac{M(f, x, r) - f(x)}{r^2}$.

20 Proof: Recall that for a C^2-function

$$2\pi r \frac{d}{dr}\left[M(f,x,r)\right] = \int_{B_r(x)} \nabla^2 f(y)\,dy$$

(i) Suppose first that $\nabla^2 f(x) \geq 0$ for $x \in D$. Then $\frac{d}{dr}\left[M(f,x,r)\right] \geq 0$ for $r \geq 0$ with $B_r(x) \subseteq D$. Thus $f(x) = \lim_{t\to 0^+} M(f,x,t) \leq M(f,x,r)$. So f is subharmonic.

(ii) $$M(f,x,r) - f(x) = \int_0^r \frac{d}{dt}\left[M(f,x,t)\right] dt$$

$$= \int_0^r \frac{1}{2\pi t}\int_{B_t(x)} \nabla^2 f(y)\,dy\,dt$$

$$= \frac{1}{4}\int_0^r 2t \; \frac{1}{\pi t^2}\int_{B_t(x)} \nabla^2 f(y)\,dy\,dt$$

$$= \frac{1}{4}\int_0^r 2t\; A(\nabla^2 f, x, t)\,dt$$

Note that for any continuous function g on D

$$g(x) = \lim_{r\to 0^+} \frac{1}{r^2}\int_0^r 2tA(g,x,t)\,dt.$$

Thus $$\lim_{r\to 0^+} \frac{M(f,x,r) - f(x)}{r^2} = \frac{\nabla^2 f(x)}{4}.$$

Hence if f is subharmonic we have $\nabla^2 f(x) \geq 0$ for all $x \in D$.

The Maximum Modulus Principle for Subharmonic Functions

Suppose f is sub-harmonic on $\mathbb{R}^2$, $B(x,r) = \{y \in \mathbb{R}^2 \mid \|x-y\| < r\}$. dy = Lebesgue measure on $\mathbb{R}^2$ $= r\,d\theta\,dr$ in polar form

$$\int_{B(x,r)} f(y)\,dy = \int_{[0,r]} \int_0^{2\pi} f(x+te^{i\theta})\,d\theta\, t\,dt$$

$$= 2\pi \int_{[0,r]} t\, M(f,x,t)\,dt \quad (*) \quad \text{(by subharmonicity)}$$

$$\geq 2\pi \int_{[0,r]} t\, f(x)\,dt = f(x)\,\pi r^2.$$ Thus

$$f(x) \leq \frac{1}{\pi r^2} \int_{B(x,r)} f(y)\,dy.$$

If f has a local maximum at $x \in \mathbb{R}^2$ then there is $r > 0$ such that for $0 < \|x-y\| < r$ $f(x) > f(y)$. Hence $f(x) = \frac{1}{\pi r^2} \int_{B(x,r)} f(x)\,dy$

$$> \frac{1}{\pi r^2} \int_{B(x,r)} f(y)\,dy \geq f(x).$$

(*) Recall that $M(f,x,r) = \frac{1}{2\pi} \int_0^{2\pi} f(x+re^{i\theta})\,d\theta$

(2) This contradiction shows that f cannot have a local maximum.

Let $K \subseteq \mathbb{R}^2$ be compact. For each $x \in K$ there is $r_x > 0$ such that $f(y) \leq 1 + f(x)$ for all y such that $\|x - y\| < r_x$. By the compactness of K there are a finite number of x's, say $x_1, .., x_n$ such that

$$K \subseteq \bigcup_{i=1}^{n} B(x_i, r_{x_i}) .$$

Hence for $y \in K$

$$f(y) \leq 1 + \max\{f(x_1), .., f(x_n)\}.$$

So f is bounded above on K. Let $\alpha = \sup\{f(x) \mid x \in K\}$. Then $\exists \{x_n\}_n \subseteq K$ with $f(x_n) \to \alpha$. Since K is compact, we may assume without loss of generality (by passing to a subsequence, if necessary) that $x_n \to x \in K$. By upper semi-continuity $\alpha = \overline{\lim_{n\to\infty}} f(x_n) \leq f(x)$. Since α is the sup we have $f(x) = \alpha$.

Conclusion: For every $x \in \mathbb{R}^2$ and $r > 0$ f attains its maximum value on the boundary of $\overline{B(x, r)}$.

The Laplacian of $\lambda \mapsto L_\varepsilon(a-\lambda)$

Let (M, τ) be a finite von Neumann algebra with faithful normal trace τ. Suppose $a \in M$, $\mathcal{U} \subseteq \mathbb{C}$ is open and $K \in \mathbb{R}$ is such that

$\|a_\lambda\|^2 < K$ for $\lambda \in \overline{\mathcal{U}}$. $a_\lambda = a - \lambda$.

Define $L, R : M \longrightarrow M$ by $L(b) = a_\lambda^* a_\lambda b$ and $R(b) = b a_\lambda a_\lambda^*$. L & R are bounded and commute

$$\frac{\partial}{\partial \lambda} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{\lambda}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\nabla^2 = 4\frac{\partial^2}{\partial \bar{\lambda}\partial \lambda}, \quad \frac{\partial}{\partial \lambda} a_\lambda = -1, \quad \frac{\partial}{\partial \lambda} a_\lambda^* = 0$$

$$\frac{\partial}{\partial \lambda} a_\lambda^* a_\lambda = -a_\lambda^*, \quad \frac{\partial}{\partial \bar{\lambda}} a_\lambda^* a_\lambda = -a_\lambda .$$

Lemma 1 (i) $\displaystyle \frac{\partial}{\partial \lambda} \tau((a_\lambda^* a_\lambda)) = -n\,\tau((a_\lambda^* a_\lambda)^{n-1} a_\lambda^*)$

(ii) $\displaystyle \frac{\partial}{\partial \bar{\lambda}} \tau((a_\lambda^* a_\lambda)^n a_\lambda^*) = -\sum_{k=0}^{n} \tau((a_\lambda^* a_\lambda)^k (a_\lambda a_\lambda^*)^{n-k})$

Lemma 2 Let $f_n(t) = t^n$. Then

$$\frac{\partial^2}{\partial \bar{\lambda}\partial \lambda} \tau(f_n(a_\lambda^* a_\lambda)) = -\tau\left(\frac{L f_n'(L) - R f_n'(R)}{L - R}(1)\right)$$

(22)

Proof $\frac{\partial^2}{\partial\bar\lambda\partial\lambda}\tau(f_n(a_\lambda^* a_\lambda))$

$$= -n\sum_{k=0}^{n-1}\tau((a_\lambda^* a_\lambda)^k (a_\lambda a_\lambda^*)^{n-k-1})$$

$$= -n\sum_{k=0}^{n}\tau(L^k R^{n-k-1}(1))$$

$$= -n\,\tau\left(\frac{L^n-R^n}{L-R}(1)\right) = -\tau\left(\frac{Lf_n'(L)-Rf_n'(R)}{L-R}(1)\right)$$

Remark If $a \geq 0$ then $\|a\| < K$

$\Rightarrow 0 \leq a \leq K \Rightarrow 0 \leq K-a \leq K \Rightarrow \|K-a\| \leq K$.

Suppose $a \notin \mathbb{C}1 \subseteq M$. Then $\|a_\lambda\| > 0$ for all $\lambda \in \mathbb{C}$.

Let $\delta_0 = \min\limits_{|\lambda| \leq 3\|a\|/2} \|a_\lambda\|^2$. If $|\lambda| > \frac{3\|a\|}{2}$ then

$\|a_\lambda\| \geq \frac{1}{2}\|a\|$. So $\|a_\lambda\|^2 \geq \delta = \min\{\delta_0, \frac{\|a\|^2}{4}\}$

for all λ. Recall that we have

chosen K such that $\|a_\lambda\|^2 \leq K-\delta$

for $\lambda \in \overline{\mathcal{U}}$. Then $0 < \delta < a_\lambda^* a_\lambda < K-\delta$,

so $0 \leq K - a_\lambda^* a_\lambda \leq K-\delta$ for $\lambda \in \overline{\mathcal{U}}$ so

$\|K - a_\lambda^* a_\lambda\| \leq K-\delta$ for $\lambda \in \overline{\mathcal{U}}$.

Lemma 4 Let $f(x) = (x-K)^n$. Then

$$\left\|\frac{Lf'(L)-Rf'(R)}{L-R}(1)\right\| < (2n^2+n)K\|a_\lambda^* a_\lambda\|^{n-1}$$

Proof: $f'(x) = n(x-K)^{n-1}$

$$\frac{Lf'(L)-Rf'(R)}{L-R} = n\,\frac{L(L-K)^{n-1}-R(R-K)^{n-1}}{(L-K)-(R-K)}$$

$$= n\,\frac{(L-K)^n-(R-K)^n}{(L-K)-(R-K)} + nK\,\frac{(L-K)^{n-1}-(R-K)^{n-1}}{(L-K)-(R-K)}$$

$$= n\sum_{k=0}^{n}(L-K)^k(R-K)^{n-k} + nK\sum_{k=0}^{n-1}(L-K)^k(R-K)^{n-k-1}$$

$$\left\|\frac{Lf'(L)-Rf'(R)}{L-R}\right\| \le n\sum_{k=0}^{n}\|a_\lambda^* a_\lambda - K\|^k\,\|a_\lambda a_\lambda^* - K\|^{n-k}$$

$$+ nK\sum_{k=0}^{n-1}\|a_\lambda^* a_\lambda - K\|^k\,\|a_\lambda a_\lambda^* - K\|^{n-k-1}$$

$$\le n(n+1)(K-\delta)^n + Knn(K-\delta)^{n-1}$$

$$= (K-\delta)^{n-1}\{n(n+1)(K-\delta) + n^2K\}$$

$$\le n(n+1)(K-\delta)^{n-1}\{K-\delta+K\}$$

$$\le 2n(n+1)(K-\delta)^{n-1}K.$$

Theorem Suppose $0<\delta<a_\lambda^* a_\lambda < K-\delta$ for $\lambda \in \mathcal{U}$, and $f(z) = \sum_{n=0}^{\infty}\alpha_n(z-K)^n$ has radius of convergence at least K. Let $p_n(z) = \sum_{k=0}^{n}\alpha_k(z-K)^k$ then

$$\left\{\frac{Lp_n'(L) - Rp_n'(R)}{L-R}\right\}_{n=1}^{\infty}$$ converges in norm. If T is the limit then

$$(L-R)T = Lf'(L) - Rf'(R).$$

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Proof: Note that $\sum_{n=1}^{\infty} |\alpha_n| (2n^2+2n) K (K-d)^{n-1} < \infty$

So $\sum |\alpha_n| \, \| \frac{L f_n'(L) - R f_n'(R)}{L-R}(1) \| < \infty$.

Hence $\left\{ \frac{L p_n'(L) - R p_n'(R)}{L-R}(1) \right\}_{n=1}^{\infty}$ for $f_n = (x-K)^n$

converges in norm. Let T be the

limit then $(L-R)T = \lim_n L p_n'(L) - R p_n'(R)(1)$

$= L f'(L) - R f'(R)(1)$.

Hence $T = \frac{L f'(L) - R f'(R)}{L-R}(1)$

Corollary $\frac{\partial^2}{\partial \bar\lambda \partial \lambda} \tau(f(a_\lambda^* a_\lambda)) = -\tau\left(\frac{L f'(L) - R f'(R)}{L-R}(1) \right)$

Proof: The convergence of $p_n(a_\lambda^* a_\lambda)$ to $f(a_\lambda^* a_\lambda)$

is uniform in $\overline{U}$ so we may differentiate

term by term. Thus $\frac{\partial^2}{\partial \bar\lambda \partial \lambda} \tau(f(a_\lambda^* a_\lambda))$

$= \frac{\partial}{\partial \bar\lambda \partial \lambda} \lim_n \tau(p_n(a_\lambda^* a_\lambda)) = \lim_n \frac{\partial^2}{\partial \bar\lambda \partial \lambda} \tau(p_n(a_\lambda^* a_\lambda))$

$= \lim_n -\tau\left(\frac{L p_n'(L) - R p_n'(R)}{L-R}(1) \right) = -\tau\left(\frac{L f'(L) - R f'(R)}{L-R}(1) \right)$

Example Let $\varepsilon > 0$ and $f(x) = \log(x+\varepsilon)$. Recall that for any $c>0$ we can write

$$\log(x) = \log c + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n c^n}(x-c)^n,$$ with radius

of convergence c. Choose K so that
$\|a_\lambda^* a_\lambda\| < K - \varepsilon$ for $\lambda \in \overline{U}$. Then
$\|a_\lambda^* a_\lambda - K\| < K < K+\varepsilon$ for $\lambda \in \overline{U}$.

Let $c = K+\varepsilon$. Then

$$f(x) = \log(x+\varepsilon) = \log(K+\varepsilon) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(K+\varepsilon)^n}(x+\varepsilon-(K+\varepsilon))^n$$

$$= \log(K+\varepsilon) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(K+\varepsilon)^n}(x-K)^n.$$

$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(K+\varepsilon)^{n+1}}(x-K)^n$. So $f'(L)$ & $f'(R)$

converge as $\|L\| = \|L(1)\| = \|a_\lambda^* a_\lambda\| < K < K+\varepsilon$.

So let us compute $[Lf'(L) - Rf'(R)](b)$

$= L(L+\varepsilon)^{-1}(b) - R(R+\varepsilon)^{-1}(b)$

$= (L+\varepsilon)^{-1}(R+\varepsilon)^{-1}[L(R+\varepsilon)(b) - R(L+\varepsilon)(b)]$

$= (L+\varepsilon)^{-1}(R+\varepsilon)^{-1}\varepsilon(L-R)(b)$

$= (L-R)(L+\varepsilon)^{-1}(R+\varepsilon)^{-1}\varepsilon(b)$. Thus

$$\frac{Lf'(L) - Rf'(R)}{L-R}(b) = \varepsilon(L+\varepsilon)^{-1}(R+\varepsilon)^{-1}(b)$$

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Theorem Let $a \in M$ and $f(\lambda) = L(a-\lambda)$. Then f is subharmonic

Proof: Let $f_\varepsilon(\lambda) = L_\varepsilon(a-\lambda) = \frac{1}{2} L(a_\lambda^* a_\lambda + \varepsilon)$

$$= \frac{1}{2}\tau(\log(a_\lambda^* a_\lambda + \varepsilon))$$

$$\nabla^2 f_\varepsilon(\lambda) = 4\frac{\partial^2}{\partial\bar\lambda\partial\lambda} f_\varepsilon(\lambda) = 2\frac{\partial^2}{\partial\bar\lambda\partial\lambda}\tau(\log(a_\lambda^* a_\lambda + \varepsilon))$$

$$= 2\varepsilon\tau((a_\lambda^* a_\lambda + \varepsilon)^{-1}(a_\lambda a_\lambda^* + \varepsilon)^{-1}) \geq 0$$

because $(a_\lambda a_\lambda^* + \varepsilon)^{-1/2}(a_\lambda^* a_\lambda + \varepsilon)^{-1}(a_\lambda a_\lambda^* + \varepsilon)^{-1/2} \geq 0$.

Thus f_ε is subharmonic on $\mathbb{R}^2$ for all $\varepsilon > 0$

We saw that $\lim_{\varepsilon \to 0} f_\varepsilon(\lambda) = L(a-\lambda)$ pointwise. Hence f is decreases subharmonic on $\mathbb{R}^2$.

Convexity of Subharmonic functions

Suppose $D \subseteq \mathbb{C}$ is open $f : D \to [-\infty, \infty)$ is subharmonic. Let $A_{r_1,r_2}(x)$ $= B_{r_2}(x) \setminus \overline{B_{r_1}(x)}$, for $0 < r_1 < r_2$.

Suppose $A_{r_1,r_2}(x) \subseteq D$. Then $r \longmapsto M(f,x,r)$ is a convex function of $\log(r)$, for $r_1 < r < r_2$.

Proof: Suppose f is C^2. Then

$$0 \le \int_{A_{r_1, r_2}(x)} \nabla^2 f(y)\, dy = \int_{\partial A_{r_1, r_2}(x)} \nabla f \cdot \vec{n}\, ds$$

$$= \int_{\partial B_{r_2}(x)} \frac{\partial f}{\partial r}\, ds - \int_{\partial B_{r_1}(x)} \frac{\partial f}{\partial r}\, ds$$

$$= 2\pi \left[r_2 \frac{d}{dr} M(f, x, r_2) - r_1 \frac{d}{dr} M(f, x, r_1) \right]$$

Thus $u(r) = 2\pi r \frac{d}{dr} M(f, x, r)$ is an increasing function of r. Let $v = \log(r)$

Then $\frac{dM}{dv}(r) = \frac{dM}{dr} \cdot \frac{dr}{dv} = r \frac{dM}{dv}$. Thus

$2\pi \frac{dM}{dv}$ is increasing and so M is an increasing convex function of v. So for $r < s < t$

$$\frac{M(f, x, s) - M(f, x, r)}{\log(s) - \log(r)} \le \frac{M(f, x, t) - M(f, x, s)}{\log t - \log s}.$$

This passes, by MCT, to decreasing sequences of functions and hence for all subharmonic functions.

If $\frac{dM}{dr}(r)$ exists we let

$$\mu_f(B_r(x)) = 2\pi r \frac{dM}{dr}(r)$$ which

exists a.e. μ_f is the Riesz measure of f.

If μ is a regular Borel measure on D and $v \in C_0^\infty(D)$ then $\int_D v(y)\,d\mu(y) = \int_D f(x)\nabla^2 v(y)\,dy$

where $f(x) = \frac{1}{2\pi}\int_D \log|x-y|\,d\mu(y)$

Theorem (Riesz 19

Suppose f is subharmonic on D. Then $\exists\,\mu_f$ a regular Borel measure on D and $E \subseteq D$ compact such that

$$f(x) = \int_E \log|x-y|\,d\mu(y) + h(x)$$

with h harmonic on E°.

The Laplacian of $\tau(\log(a_\lambda^* a_\lambda + \varepsilon))$: $a_\lambda = a - \lambda$

Suppose $\mathcal{U} \subseteq \mathbb{C}$ is open and $\lambda \in \mathcal{U}$,

we want to compute $\frac{\partial^2}{\partial\bar\lambda\partial\lambda} \tau(\log(a_\lambda^* a_\lambda + \varepsilon))$

for $\lambda \in \mathcal{U}$ and show that it's positive for all $\varepsilon > 0$. We are interested in the

case when $\varepsilon \searrow 0$. But we start withe the case $\varepsilon \in \mathbb{C}$, $Re(\varepsilon) > 0$ and $|\varepsilon| > \|a_\lambda^* a_\lambda\|$ for $\lambda \in \overline{\mathcal{U}}$. Then

$$\tau(\log(a_\lambda^* a_\lambda + \varepsilon)) = \log(\varepsilon) + \tau(\log(1 + \varepsilon^{-1} a_\lambda^* a_\lambda))$$

$$= \log(\varepsilon) + \tau\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\varepsilon^n} (a_\lambda^* a_\lambda)^n\right)$$

$$= \log(\varepsilon) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\varepsilon^n} \tau((a_\lambda^* a_\lambda)^n).$$

The series converges uniformly on $\overline{\mathcal{U}}$ so we may differentiate term by term to

conclude $\frac{\partial^2}{\partial\bar\lambda\partial\lambda} \tau(\log(a_\lambda^* a_\lambda + \varepsilon))$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\varepsilon^n} \frac{\partial^2}{\partial\bar\lambda\partial\lambda} \tau((a_\lambda^* a_\lambda)^n)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\varepsilon^n} \sum_{k=0}^{n-1} \tau\left((a_\lambda^* a_\lambda)^k (a_\lambda a_\lambda^*)^{n-k-1}\right)$$

$$= \varepsilon^{-1} \sum_{n=1}^{\infty} \sum_{\substack{k,l \geq 0 \\ k+l=n-1}} \tau\left((-\varepsilon^{-1} a_\lambda^* a_\lambda)^k (-\varepsilon^{-1} a_\lambda a_\lambda^*)^l\right)$$

$$= \varepsilon^{-1} \sum_{n=0}^{\infty} \sum_{\substack{k,l \geq 0 \\ k+l=n}} \tau\left((-\varepsilon^{-1} a_\lambda^* a_\lambda)^k (-\varepsilon^{-1} a_\lambda a_\lambda^*)^l\right)$$

$$= \varepsilon^{-1} \sum_{k,l \geq 0} \tau\left((-\varepsilon^{-1} a_\lambda^* a_\lambda)^k (-\varepsilon^{-1} a_\lambda a_\lambda^*)^l\right)$$

$$= \varepsilon^{-1} \tau\left((1+\varepsilon^{-1} a_\lambda^* a_\lambda)^{-1} (1+\varepsilon^{-1} a_\lambda a_\lambda^*)^{-1}\right)$$

$$= \varepsilon \tau\left((a_\lambda^* a_\lambda + \varepsilon)^{-1} (a_\lambda a_\lambda^* + \varepsilon)^{-1}\right).$$

Thus $\nabla^2 \tau(\log(a_\lambda^* a_\lambda + \varepsilon))$

$$= 4\varepsilon \tau\left((a_\lambda^* a_\lambda + \varepsilon)^{-1} (a_\lambda a_\lambda^* + \varepsilon)^{-1}\right).$$

Hence $\nabla^2 L_\varepsilon(a-\lambda) = \frac{1}{2} \nabla^2 \tau(\log(a_\lambda^* a_\lambda + \varepsilon))$

$$= 2\varepsilon \tau\left((a_\lambda^* a_\lambda + \varepsilon)^{-1} (a_\lambda a_\lambda^* + \varepsilon)^{-1}\right)$$

The right hand side is an analytic function of ε for $\mathrm{Re}(\varepsilon) > 0$. Let us show that the left hand side is also analytic function of ε for $\mathrm{Re}(\varepsilon) > 0$. Having done this the equality holds for all

$\varepsilon \in \mathbb{C}$ with $Re(\varepsilon) > 0$. We shall show then $\exists\ K, \delta > 0$ so that for $\lambda \in \overline{\mathcal{U}}$ $\varepsilon \in \overline{V}$ we have $\| a_\lambda^* a_\lambda + \varepsilon - K \| < K - \delta$

Then $\tau(\log(a_\lambda^* a_\lambda + \varepsilon)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n K^n} \tau((a_\lambda^* a_\lambda + \varepsilon - K)^n)$

converges uniformly on $\overline{\mathcal{U}} \times \overline{V}$. Thus we can apply $\frac{\partial^2}{\partial \bar{\lambda} \partial \lambda}$ term by term and each term will be a polynomial in ε. Thus the series will be analytic in ε as required.

$\sigma(a_\lambda^* a_\lambda) \subseteq [0, M]$ $\lambda \in \overline{\mathcal{U}}$

$\sigma(a_\lambda^* a_\lambda + \varepsilon) \subseteq Re(\varepsilon) + [0, M]$
$+ i\, Im(\varepsilon)$