Freeness and R-diagonal Operators

non-commutative probability space : $(\mathcal{A}, \varphi)$

$\mathcal{A} =$ cx. unital algebra, $\varphi : \mathcal{A} \xrightarrow{\text{linear}} \mathbb{C}$, $\varphi(1) = 1$.

e.g. $\mathcal{A} = \mathbb{C}[\mathbb{F}_n]$ $\quad f * g(\gamma) = \sum_{\sigma} f(\sigma) g(\sigma^{-1}\gamma)$

$\varphi(f) = f(e)$

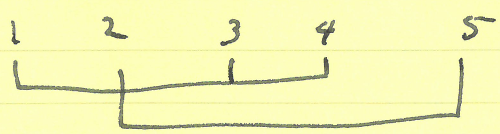
$\mathcal{A}_1, \ldots, \mathcal{A}_s \subseteq \mathcal{A}$ unital subalgebras are freely independent if $\forall\, a_1, \ldots, a_n \in \mathcal{A}$ with $\varphi(a_i) = 0$ and $a_i \in \mathcal{A}_{j_i}$ with $j_1 \neq j_2, j_2 \neq j_3, \ldots, j_{n-1} \neq j_n$ we have $\varphi(a_1 \cdots a_n) = 0$.

elements $a_1, \ldots a_s$ are freely independent if the unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are freely independent where $\mathcal{A}_i = \mathrm{alg}(1, a_i)$

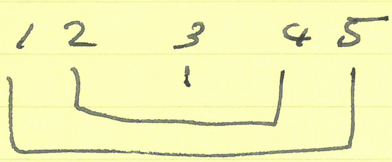
Example $\mathbb{F}_n = \langle \gamma_1, \ldots, \gamma_n \rangle$ $\quad \mathcal{A}_i = \mathrm{alg}(1, \gamma_i, \gamma_i^{-1})$ are freely independent.

Partitions & Cumulants

$P(n) = \{\text{partitions of } [n]\}$, $[n] = \{1, 2, \ldots, n\}$



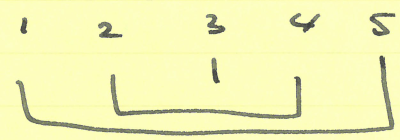
crossing



non-crossing

$NC(n) = \{\pi \in P(n) \mid \pi \text{ is non-crossing}\}$

② $\pi \le \sigma$ means every block of π is contained in some block of σ. 1_n = partition with 1 block. $\pi \vee \sigma$ = smallest partition larger than both π & σ.

If $a_1, \dots, a_5 \in (\mathcal{A}, \varphi)$, $\pi =$ 

$$\varphi_\pi(a_1, \dots, a_5) = \varphi(a_1 a_5)\, \varphi(a_2 a_4)\, \varphi(a_3).$$

$$K_1(a_1) = \varphi(a_1)$$

$$K_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\, \varphi(a_2)$$

$$K_3(a_1, a_2, a_3) = \varphi(a_1 a_2 a_3) - \{\varphi(a_1)\, \varphi(a_2 a_3) + \varphi(a_2)\, \varphi(a_1 a_3) + \varphi(a_1 a_2)\, \varphi(a_3)\} + 2\, \varphi(a_1)\, \varphi(a_2)\, \varphi(a_3)$$

In general
$$\kappa_n(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} \mu(\pi, 1_n) \times \varphi_\pi(a_1, \dots, a_n). \qquad (*)$$

where $\mu : NC(n) \times NC(n) \to \mathbb{Z}$ is the Möbius function of $NC(n)$. There is an explicit formula for μ but an easier way to write $(*)$ is

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \dots, a_n)$$

where
$$\kappa_\pi(a_1, \dots, a_n) = \prod_{\substack{V \in \pi \\ V = (i_1, \dots, i_k)}} \kappa_k(a_{i_1}, \dots, a_{i_k})$$

Fact if $a_i = 1$ for some i $\kappa_n(a_1, \dots, a_n) = 0$

Theorem (Nica – Speicher)

$a_1, \dots a_s \in (\mathcal{A}, \varphi)$ are free if and only if mixed cumulants vanish; this means that for all $i_1 \dots i_n \in [s]$ $\kappa_n(a_{i_1}, \dots, a_{i_n}) = 0$ unless $i_1 = \dots = i_n$.

R-transform Let $a \in (\mathcal{A}, \varphi)$ and $\kappa_n = \kappa_n(a, \dots, a)$. These are the free cumulants of a. Let $R(z) = \kappa_1 + \kappa_2 z + \cdots$ as a formal power series. Let $G(z)$ $= \sum_{n \geq 0} \frac{\varphi(a^n)}{z^{n+1}}$ also as a formal power series

then $G\left(\frac{1}{z} + R(z)\right) = z = \frac{1}{G(z)} + R(G(z))$.

If $a \in (M, \tau)$ $a = a^*$ $G(z) = \varphi((z-a)^{-1})$ for $z \in \mathbb{C}^+$ Then R is analytic and series converges.

If a is a semi-circle then $R(z) = z$.

Product Rule

Given $r \geq 2$ and $n_1 + \dots + n_r = n$, $a_1 \dots a_n \in \mathcal{A}$

$$\kappa_r(a_1 \cdots a_{n_1}, a_{n_1+1} \cdots a_{n_1+n_2}, \dots, a_{n_1+\cdots+n_{r-1}+1} \cdots a_{n_1+\cdots+n_r})$$

$$= \sum_{\substack{\pi \in NC(n) \\ \pi \vee \rho = 1_n}} \kappa_\pi(a_1, \dots, a_n) \text{ where } \rho = \{(1, \dots, n_1), (n_1+1, \dots, n_1+n_2), \dots, (n_1+\cdots+n_{r-1}+1, \dots, n_1+\cdots+n_r)\}.$$

④ **Example** $\kappa_{n-1}(a_1a_2, a_3, \dots, a_n) = \kappa_n(a_1, \dots, a_n)$

$$+ \sum_{j=1}^{n-1} \kappa_{n-j}(a_1, a_{j+2}, \dots, a_n)\, \kappa_j(a_2, \dots, a_{j+1})$$

because $\pi \vee \sqcup | | \cdots | = 1_n$ means either $\pi = 1_n$ or π has 2 blocks with 1 in one block and 2 in the other.

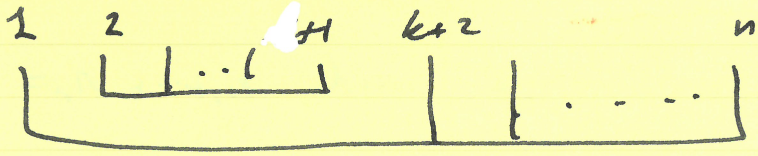
Example Bernoulli Operator

$$b = b^*, \; b^2 = 1, \; \varphi(b) = 0.$$

$$0 = \kappa_{n-1}(1, b, \dots, b) = \kappa_{n-1}(b^2, b, \dots b)$$

$$(*) \quad = \sum_{\substack{\pi \in NC(n) \\ \pi \vee \rho = 1_n}} \kappa_\pi(b, \dots, b) \qquad \rho = \sqcup | | \cdots | .$$

Now if $\pi \vee \rho = 1_n$ then either π has 1 block i.e. $\pi = 1_n$ or π has two blocks and 1 & 2 are in different blocks: so for some j

$\pi =$ 

$$\kappa\ (b, \dots b) = \kappa_k(b, \dots, b)\, \kappa_{n-k}(b, \dots b)$$

So let $\alpha_m = (-1)^m \kappa_{2(m+1)}(b, \dots b)$

Then $(*) \Rightarrow \alpha_{m-1} = \sum_{\ell=1}^{m-1} \alpha_\ell \, \alpha_{m-\ell-1}$. But this is exactly the recursion of the Catalan numbers; so $\alpha_m = \frac{1}{m+1}\binom{2m}{m}$ and

thus $$\kappa_n(b,\dots,b) = \begin{cases} 0 & n \text{ odd} \\ (-1)^{m-1} C_{m-1} & n = 2m \end{cases}$$

R-diagonal Operators

$a \in (M, \varphi)$ is R-diagonal if $\forall\, \varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ $\kappa_n(a^{(\varepsilon_1)}, \dots, a^{(\varepsilon_n)}) = 0$ unless

- n is even
- $\varepsilon_i = -\varepsilon_{i+1}$ for $1 \le i < n-1$.

Example

$c = \frac{1}{\sqrt{2}}(x_1 + i x_2)$ is a circular with x_1 & x_2 free & semi-circular.

$$\kappa_n(c^{(\varepsilon_1)}, \dots, c^{(\varepsilon_n)}) = \kappa_n(x_1 + i\varepsilon_1 x_2, x_1 + i\varepsilon_2 x_2, \dots, x_1 + i\varepsilon_n x_n) 2^{-n/2}$$
$$= 2^{-n/2}\{\kappa_n(x_1, \dots, x_1) + (i)^n \varepsilon_1 \cdots \varepsilon_n \kappa_n(x_2 \dots x_2)\}.$$

So $\kappa_n(c^{(\varepsilon_1)} \dots c^{(\varepsilon_n)}) = 0$ for $n \neq 2$. When $n = 2$

$$\kappa_2(c^{(\varepsilon_1)}, c^{(\varepsilon_2)}) = 2^{-1}\{1 + i^2 \varepsilon_1 \varepsilon_2\} = \begin{cases} 0 & \varepsilon_1 = \varepsilon_2 \\ 1 & \varepsilon_1 \neq \varepsilon_2 \end{cases}$$

⑥ Hence c is R-diagonal and the only non-vanishing cumulants are

$$\kappa_2(c, c^*) = \kappa_2(c^*, c) = 1.$$

Theorem Let a be R-diagonal and

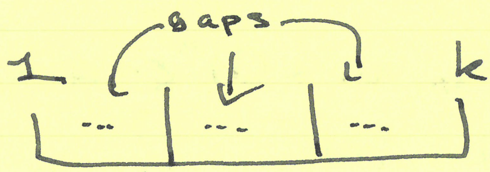
$$\alpha_n = \kappa_{2n}(a, a^*, \dots, a, a^*) = \kappa_{2n}(a^*, a, \dots, a^*, a).$$

Then $\kappa_n(a^*a, \dots, a^*a) = \sum_{\pi \in NC(n)} \alpha_\pi$. In particular the $*$-distribution of a is determined by the distribution of the positive operator a^*a.

Proof: Let $\rho = \{(1,2),(3,4),\dots,(2n-1,2n)\} \in NC(2n)$.

Then $\kappa_n(a^*a, \dots, a^*a) = \sum_{\substack{\pi \in NC(2n) \\ \pi \vee \rho = 1_{2n} \ \& \ \kappa_\pi(a^*, a, \dots, a^*, a) \neq 0}} \kappa_\pi(a^*, a, \dots, a^*a)$.

Let $\pi \in NC(2n)$ with $\pi \vee \rho = 1_{2n}$. Let $V \in \pi$. Suppose $2k+1 \in V$. Claim $2k \in V$. For simplicity suppose $1 \in V$ and we'll show $2n \in V$. Let k be the last element of V. Then $|V| + |gaps| = k$ where $|gaps|$ is the sum of the gaps between 1 and k



. So k is even. If $k < 2n$ then the blocks $(k+1, k+2) \dots (2n-1, 2n)$ of ρ are not connected by π so

$\pi \vee \rho < 1_{2n}$. Thus for every k

$2k+1$ & $2k$ are in the same block of π.

Now let $\sigma \in NC(n)$ and $V = (i_1 \ldots i_k) \in \sigma$.

Let $W = (2i_1-1, 2i_1, \ldots 2i_k-1, 2i_k)$. Applying

this to all blocks of σ we get π a

non-crossing partition of $[2n]$ with $\pi \vee \rho = 1_{2n}$.

Also $\kappa_\pi(a^*a, \ldots, a^*a) = \alpha_\sigma$. Thus the

bijection gives us that

$$\kappa_n(a^*a, \ldots, a^*a) = \sum_{\sigma \in NC(n)} \alpha_\sigma .$$

<u>Theorem</u>

Suppose b & x are even and free

with b Bernoulli. Then $a = bx$ is R-diagonal

and $\varphi((a^*a)^n) = \varphi(x^{2n})$.

<u>Proof</u>: Let $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$, and suppose

$\kappa_n(a^{(\varepsilon_1)}, \ldots, a^{(\varepsilon_n)}) \neq 0$. Let us show first

that n is even. By the product formula

$\kappa_n(a^{(\varepsilon_1)} \ldots a^{(\varepsilon_n)})$

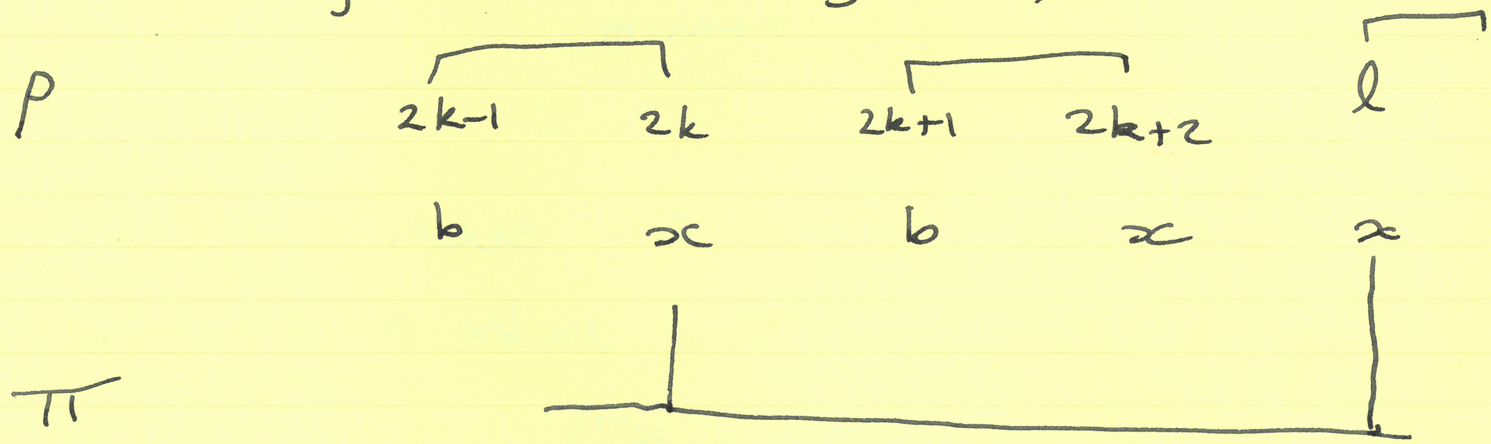
$$= \sum_{\substack{\pi \in NC(2n) \\ \pi \vee \rho = 1_{2n}}} \kappa_\pi(b, x, \ldots \qquad)$$

↑ supposing $\varepsilon_1 = 1$.

(8) Suppose $\pi \in NC(2n)$ and $\kappa_\pi(\ldots, \ldots, \ldots) \neq 0$.

Because x & b are free π must consist of b-blocks & x-blocks. There are n b's and n x's. So the lengths of the b blocks is n. But all b-blocks are even thus n is even.

Now suppose $\varepsilon_1 \ldots \varepsilon_n \in \{-1, 1\}$, and for some k $\varepsilon_k = \varepsilon_{k+1}$. Let V be the block of π containing $2k$, and let



l be the next element of V. The gap between $2k$ & l must consist of x-blocks & b-blocks and thus l is odd. Thus the gap between $2k$ & l is not connected by π. The only possibility is that either $l = 2k+1$, which is impossible because $2k+1$ is a "b", or that

$2k$ is the last element of V. In this case let l be the first element. Apply the previous argument to the cyclical gap from $2k$ to l; and again we get a contradiction. Thus there can be no k such that $\varepsilon_k = \varepsilon_{k+1}$. Hence $a = bx$ is R-diagonal. $\square$

Remark

Suppose $x \geqslant 0$. Let $y = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$ and b be Bernoulli free from y. Then $a = by$ is R-diagonal the distribution of $a^*a = \begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix}$ is the same as x^2.

So given any $x \geqslant 0$ there is an R-diagonal element a such that a^*a and x^2 have the same distribution.