The S-transform

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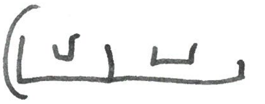
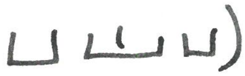
$a \in (\mathcal{A}, \varphi)$ $\qquad m_n = \varphi(a^n)$

$$\kappa_n = \kappa_n(a, \dots, a)$$

(*) $\qquad m_n = \sum_{\pi \in NC(n)} \kappa_\pi \qquad \kappa_n = \sum_{\pi \in NC(n)} \mu(\pi, 1_n)\, m_\pi$

$\mu(\pi, 1_n) \in \mathbb{Z}$ $\qquad \mu(\pi, 1_n)$ is always a product of signed Catalan numbers

$\beta_n = (-1)^{n+1} C_{n-1} \qquad \mu(\pi, 1$

$\mu\Big($  $, 1_n\Big) = \mu\big(0_n,$  $\big)$

$$= \beta_2 \beta_2 \beta_1 \beta_2$$

$$M(z) = 1 + \sum_{n=1}^{\infty} m_n z^n = \sum_{n=0}^{\infty} m_n z^n \qquad m_0 = 1,\ \kappa_0 = 1$$

$$C(z) = 1 + \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=0}^{\infty} \kappa_n z^n$$

(*) $\Rightarrow \qquad M(z) = C(zM(z))$

(*) $\Rightarrow \qquad m_n = \sum_{s=1}^{n} \sum_{\substack{i_1, \dots, i_s \geq 0 \\ s + i_1 + \dots + i_s = n}} \kappa_s m_{i_1} \cdots m_{i_s}$

$\Rightarrow \qquad M(z) = C(zM(z)). \qquad$ (**)

Let $\tilde{M}(z) = M(z) - 1$. $\tilde{R}(z) = C(z) - 1$. Then

(**) becomes $\tilde{M}(z) = \tilde{R}(f(z))$ where

$$f(z) = z(1 + \tilde{M}(z)) = z + m_1 z^2 + \dots$$

$$\tilde{R}(z) = \kappa_1 z + \kappa_2 z^2 + \dots \qquad , \tilde{M}_1(z) = m_1 z + m_2 z^2 + \dots$$

(B) Inversion of Formal Power Series

Suppose $f(z) = \alpha_1 z + \alpha_2 z^2 + \cdots$ is a formal power series with $f(0) = 0$ & $f'(0) = \alpha \neq 0$

Then we can find a formal power series

$$g(z) = \beta_1 z' + \beta_2 z^2 + \cdots \quad \text{with}$$

$$g(0) = 0 \quad \text{and} \quad g'(0) = \beta_1 = \alpha_1^{-1}$$

and $f(g(z)) = z = g(f(z))$

$$z = f(g(z)) = \alpha_1(\beta_1 z + \beta_2 z^2 + \beta_3 z^3 + \cdots)$$

$$+ \alpha_2(\beta_1 z + \beta_2 z^2 + \cdots)^2 + \alpha_3(\beta_1 z + \beta_2 z^2 + \cdots)^3 + \cdots$$

$$= \alpha_1\beta_1 z + (\beta_2 + \alpha_2 \beta_1^2) z^2 + (\beta_3 + 2\alpha_2 \beta_1 \beta_2 + \alpha_3 \beta_1^3) z^3 + \cdots$$

So
$$\alpha_1 \beta_1 = 1$$

$$\alpha_1 \beta_2 + \alpha_2 \beta_1 = 0$$

$$\alpha_1 \beta_3 + 2\alpha_2 \beta_1 \beta_2 + \alpha_3 \beta_1^3 = 0$$

Example $f(z) = e^z - 1 \qquad g(z) = \log(1+z)$

$$\alpha_n = \frac{1}{n!}, \qquad \beta_n = \frac{(-1)^{n+1}}{n}$$

$\alpha_1 = \beta_1 = 1$

$\alpha_2 = \frac{1}{2}, \quad \beta_2 = \frac{-1}{2}$

S-transform (Con't.)

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$\tilde{R}(z) = K_1 z + K_2 z^2 + \cdots \qquad \tilde{S} = \tilde{R}^{\langle -1 \rangle}$

$$\tilde{S}(z) = \frac{1}{K_1} z - \frac{K_2}{K_1^3} z^2 + \frac{2K_2^2 - K_1 K_3}{K_1^5} z^3 - \frac{5K_2^3 + 5K_1K_2K_3 + K_1K_4^2}{K_1^7} 5$$

$$- \frac{5K_2^3 - 5K_1K_2K_3 + K_1^2 K_4}{K_1^7} z^5$$

$$S(z) = \frac{\tilde{S}(z)}{z} = \frac{1}{K_1} - \frac{K_2 z}{K_1^3} + \frac{2K_2^2 - K_1 K_3}{K_1^7} z^2 - \cdots$$

Recall that $\tilde{M}(z) = m_1 z + m_2 z^2 + \cdots$ &

$f(z) = z(1 + \tilde{M}(z)) = z + m_1 z^2 + m_2 z^3 + \cdots$

$\tilde{f}(M^{\langle -1 \rangle}(z)) = M^{\langle -1 \rangle}(z)\,(1 + \tilde{M}(\tilde{M}^{\langle -1 \rangle}(z))) = (1+z)\,\tilde{M}^{\langle -1 \rangle}(z).$

Than the relation $\tilde{M} = \tilde{R} \circ f$

$\Rightarrow \tilde{R}^{\langle -1 \rangle} = f \circ \tilde{M}^{\langle -1 \rangle} \Rightarrow \tilde{S}(z) = (1+z)\,\tilde{M}^{\langle -1 \rangle}(z)$

and $\quad S(z) = \frac{1+z}{z} \tilde{M}^{\langle -1 \rangle}(z)$

Lemma 3.3 Let $a(s) = (1 - sa)^{-1} - M_a(s)$

$b(t) = (1 - tb)^{-1} M_b(t)$. For any $\rho \in \mathbb{C}$

Then $(1 - sa)(1 - \rho\, a(s) b(t))(1 - bt) = c_0 + c_1 a + c_2 b + c_3 ab$

where $\quad c_0 = 1 - \rho \tilde{M}_a(s)\, \tilde{M}_b(t)$

$c_1 = -s(1 - \rho M_a(s)\, \tilde{M}_b(t))$

$c_2 = -t(1 - \rho \tilde{M}_a(s)\, M_b(t))$

$c_3 = st(1 - \rho M_a(s)\, M_b(t))$

(D) **Proof:** $(1-sa)(1-\rho\, a(s)\, b(t))(1-tb)$

$$= (1-sa)(1-tb) - \rho\, (1-sa)\left[(1-sa)^{-1} - M_a(s)\right]\left[(1-tb)^{-1} - M_b(t)\right](1-tb)$$

$$= (1-sa)(1-tb) - \rho\left[1 - (1-sa)M_a(s)\right]\left[1-(1-tb)M_b(t)\right]$$

$$= (1-sa)(1-tb) - \rho\left[sM_a(s)a - \tilde{M}_a(s)\right]\left[tM_b(t)b - \tilde{M}_b(t)\right]$$

$$= 1 - sa - tb + st\,ab$$
$$- \rho\,\tilde{M}_a(s)\tilde{M}_b(t) + \rho\, t\,\tilde{M}_a(s) M_b(t)\, b$$
$$+ \rho\, s\, M_a(s)\tilde{M}_b(t)\, a - \rho\, st\, M_a(s) M_b(t)\, ab$$

$$= \left[1 - \rho\,\tilde{M}_a(s)\tilde{M}_b(t)\right]$$
$$- s\left[1 - \rho\, M_a(s)\tilde{M}_b(t)\right]a$$
$$- t\left[1 - \rho\,\tilde{M}_a(s) M_b(t)\right]b$$
$$+ st\left[1 - \rho\, M_a(s) M_b(t)\right]ab \qquad \square$$

Theorem Let $S_a(z) = \frac{1+z}{z}\tilde{M}_a^{\langle -1\rangle}(z)$. Then $S_{ab} = S_a S_b$, when a and b are free.

Proof: We must show $M_{ab}^{\langle -1\rangle}(z) = (1+z)\,\tilde{M}_a^{\langle -1\rangle}(z)\,\tilde{M}_b^{\langle -1\rangle}(z)$

So let $s = \tilde{M}_a^{\langle -1\rangle}(z)$, $t = \tilde{M}_b^{\langle -1\rangle}(z)$ and $v = \frac{1+z}{z}st$.

Then $\tilde{M}_a(s) = \tilde{M}_b(t)$. Let $\rho = \frac{1}{M_a(s)\tilde{M}_a(s)} = \frac{1}{z(1+z)}$. Then

$$C_0 = 1 - \rho\,\tilde{M}_a(s)\tilde{M}_b(t)$$
$$= 1 - \frac{\tilde{M}_b(t)}{M_b(t)} = \frac{M_b(t) - \tilde{M}_b(t)}{M_b(t)} = \frac{1}{M_b(t)} = \frac{1}{1+z}$$

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$$C_1 = -s(1 - \rho M_a(s) \tilde{M}_b(t)) = 0$$

$$C_2 = -t(1 - \rho \tilde{M}_a(s) M_b(t)) = 0$$

$$C_3 = st\left(1 - \frac{M_a(s) M_b(t)}{M_a(s) \tilde{M}_a(s)}\right) = st\left[\frac{\tilde{M}_a(s) - M_a(s)}{\tilde{M}_a(s)}\right]$$

$$= \frac{-st}{\tilde{M}_a(s)} = \frac{-st}{Z}$$

Hence $(1-sa)(1-\rho a(s) b(t))(1-tb) = \frac{1}{M_a(s)} - \frac{st}{\tilde{M}_a(s)} ab$

$$= \frac{1}{M_a(s)}\left(1 - \frac{M_a(s)}{\tilde{M}_a(s)} st\, ab\right)$$

$$= \frac{1}{M_a(s)}\left(1 - \frac{1+Z}{Z} st\, ab\right) = \frac{1}{M_a(s)}(1 - vab).$$

Hence $(1-tb)^{-1}(1-\rho a(s) b(t))^{-1}(1-sa)^{-1}$

$$= M_a(s)(1-vab)^{-1}.$$

Note that for $n \geq 1$

$\varphi((M_a(s)+a(s))(\rho a(s) b(t))^n (M_b(t) + b(t))) = 0$
as $a(s)$ & $b(t)$ are centred and free. Hence

$$\varphi\left((1-tb)^{-1}(1-\rho a(s) b(t))^{-1}(1-sa)^{-1}\right)$$

$$= \varphi((M_a(s) + a(s))(M_b(t) + b(t)))$$

$$= M_a(s) M_b(t).$$

Thus

$$M_a(s) M_{ab}(v) = M_a(s) \varphi((1-vab)^{-1}) = M_a(s) M_b(t)$$

So $M_{ab}(v) = M_a(s) = M_b(t)$.

(F) Thus $\tilde{M}_{ab}(v) = \tilde{M}_a(s) = \tilde{M}_b(t) = z$

Hence $\tilde{M}_{ab}^{\langle -1\rangle}(z) = v$, $\tilde{M}_a^{\langle -1\rangle}(z) = s$, $\tilde{M}_b^{\langle -1\rangle}(z) = t$.

Thus $\tilde{M}_{ab}^{\langle -1\rangle}(z) = v = \frac{1+z}{z} st = \frac{1+z}{z} \tilde{M}_a^{\langle -1\rangle}(z) \cdot \tilde{M}_b^{\langle -1\rangle}(z)$,

so $$\frac{1+z}{z} \tilde{M}_{ab}^{\langle -1\rangle}(z) = \left[\frac{1+z}{z} \tilde{M}_a^{\langle -1\rangle}(z)\right]\left[\frac{1+z}{z} \tilde{M}_b^{\langle -1\rangle}(z)\right]$$

Hence $S_{ab}(z) = S_a(z)\, S_b(z)$ ▨