The beginning of the Calculation of the Brown Measure

25 June 2021

$x = x^*$ even $\quad b = b^* \quad b^2 = 1$ Bernoulli, $\lambda > 0$

$$x(s) = (1-sx)^{-1} - \varphi\left((1-sx)^{-1}\right)$$
$$= (1-sx)^{-1} - M_x(s)$$

$$b(t) = (1-t\lambda b)^{-1} - M_{\lambda b}(t)$$

$$(1-sx)(1-\rho\, x(s)\, b(t))\,(1-t\lambda b)$$
$$= c_0 + c_1 x + c_2 \lambda b + c_3 \lambda a b$$

$$c_0 = 1 - \rho\, \tilde{M}_x(s)\, \tilde{M}_{\lambda b}(t)$$

$$c_1 = -s\left(1 - \rho\, M_x(s)\, \tilde{M}_{\lambda b}(t)\right)$$

$$c_2 = -t\left(1 - \rho\, \tilde{M}_x(s)\, M_{\lambda b}(t)\right)$$

$$c_3 = st\left(1 - \rho\, M_x(s)\, M_{\lambda b}(t)\right)$$

We want to choose s, t, and ρ so that $c_0 = c_3 = 0$

Thus we must have $\dfrac{1}{\tilde{M}_x(s)\, \tilde{M}_{\lambda b}(t)} = \dfrac{1}{M_x(s)\, M_{\lambda b}(t)}$

which implies $M_x(s) + M_{\lambda b}(t) = 1$; if this is possible we let $\rho = \dfrac{1}{\tilde{M}_x(s)\, \tilde{M}_{\lambda b}(t)} = \dfrac{1}{M_x(s)\, M_{\lambda b}(t)}$.

Now $M_{\lambda b}(t) = \int (1-tw)^{-1}\, d\mu_{\lambda b}(w)$. Since b is even $M_{\lambda b}(it) =$

$$= \frac{1}{1-\lambda^2 t^2} \in (0, 1]$$

(2) Since x is even $M_x(it) = \mathcal{E}((1-itx)^{-1}) = \mathcal{E}((1+itx)^{-1})$

so $M_x(it)$ is real. Thus

$$M_x(it) = \mathrm{Re}(M_x(it)) = \mathrm{Re}\left(\int \frac{1}{1-itw}\, d\mu_x(w)\right)$$

$$= \int \frac{1}{1+t^2w^2}\, d\mu_x(w) \in (0,1]$$

Now let $v>0$ $s=iv$ & $t=\dfrac{i}{\lambda^2 v}$ then

$$M_{\lambda b}(t) = \frac{1}{1+\lambda^2\left(\frac{i}{\lambda^2 v^2}\right)} = \frac{\lambda^2 v^2}{1+\lambda^2 v^2} = 1 - M_x(s)$$

provided v is chosen so that $\lambda^2 v^2 = \dfrac{1-M_x(iv)}{M_x(iv)}$

Note that with this choice $sM_x(s) = tM_{\lambda b}(t)$. Proof: $\dfrac{s}{t}$

$$= \frac{iv}{\frac{i}{\lambda^2 v}} = \lambda^2 v^2 = \frac{1-M_x(iv)}{M_x(iv)} = \frac{1-M_x(s)}{M_x(s)} = \frac{M_{\lambda b}(t)}{M_x(s)}.$$

Proof: Suppose $v^2\lambda^2 = \dfrac{1-M_x(iv)}{M_x(iv)}$, then

$1+\lambda^2v^2 = \dfrac{1}{M_x(iv)}$ and $\dfrac{\lambda^2 v^2}{1+\lambda^2v^2} = M_x(iv) \cdot \dfrac{1-M_x(iv)}{M_x(iv)}$

$$= 1 - M_x(iv) = 1 - M_x(s).$$

Thus for a given λ we have reduced the problem to showing that there is $v>0$ such that $\lambda^2 v^2 = \dfrac{1-M_x(iv)}{M_x(iv)}$.

When this holds and we set $s=iv$ $t = \dfrac{i}{\lambda^2 v^2}$ $\rho = \dfrac{1}{M_x(s)\, M_{\lambda b}(t)}$

we get $c_1 = -s(1 - \rho M_x(s) \tilde{M}_{\lambda b}(t))$

$$= \frac{-s}{M_x(s) M_{\lambda b}(t)} \left[M_x(s) M_{\lambda b}(t) - M_x(s) \tilde{M}_{\lambda b}(t) \right]$$

$$= \frac{-s}{M_{\lambda b}(t)} = \frac{-s}{1 - M_x(s)}$$

$$c_2 = -t(1 - \rho \tilde{M}_x(s) M_{\lambda b}(t))$$

$$= \frac{-t}{M_x(s) M_{\lambda b}(t)} \left[M_x(s) M_{\lambda b}(t) - \tilde{M}_x(s) M_{\lambda b}(t) \right]$$

$$= \frac{-t}{M_x(s)} = c_1$$

Hence $(1 - sx)(1 - \rho\, x(s))\, 6(t))(1 - t\lambda b)$

$= c_1 (x + \lambda b)$ with $c_1 = \frac{-s}{M_{\lambda b}(t)} = \frac{-t}{M_x(s)}$.

Main Calculation

$a = bx$ $\quad c_1 = \frac{-s}{M_{\lambda b}(t)}$. Supposing that

$\lambda^2 V^2 = \frac{1 - M_x(iv)}{M_x(iv)}$ $\quad (\lambda, v > 0)$ we have

$$\log |c_1^{-1}| = -\log |c_1| = -\log |s| + \log | M_{\lambda b}(t)|$$

$$= -\log(v) + \log\left(\frac{\lambda^2 v^2}{1 + \lambda^2 v^2} \right)$$

$$L(1 - sx) = \int \log|1 - sw|\, d\mu_x(w) = \frac{1}{2} \int \log |1 - sw|^2\, d\mu_x(w)$$

$$= \frac{1}{2} \int \log(1 + v^2 w^2)\, d\mu_x(w)$$

④ $L(1-t\lambda b) = \frac{1}{2} L\big((1-t\lambda b)(1+t\lambda b)\big)$

$$= \frac{1}{2} L\left(1+\frac{\lambda^2}{\lambda^4 v^2}\right) = \frac{1}{2}\log\left(\frac{1+\lambda^2 v^2}{\lambda^2 v^2}\right)$$

Thus $L(a-\lambda) = L(a+\lambda) = L(bx+\lambda)$

$$= L(b(x+\lambda b)) = L(b) + L(x+\lambda b)$$

$$= L(x+\lambda b) = L\big(c_1^{-1}(1-sx)(1-\rho x(s)b(t))(1-t\lambda b)\big)$$

$$= -\log|c_1| + L(1-sx) + L(1-\rho x(s) b(t)) + L(1-t\lambda b)$$

$$= \frac{1}{2}\int \log(1+v^2w^2)\,d\mu_x(w) - \log v + \log\left(\frac{\lambda^2 v^2}{1+\lambda^2 v^2}\right)$$

$$+ \frac{1}{2}\log\left(\frac{1+\lambda^2 v^2}{\lambda^2 v^2}\right) + L(1-\rho x(s) b(t))$$

$$= \frac{1}{2}\int \log(1+v^2w^2)\,d\mu_x(w) + \frac{1}{2}\log\left(\frac{\lambda^2}{\lambda^2 v^2+1}\right)$$

$$+ L(1-\rho x(s) b(t))$$

Lemma For $x(s) = (1-sx)^{-1} - \varphi((1-sx)^{-1})$

$b(t) = (1-t\lambda b)^{-1} - \varphi((1-t\lambda b)^{-1})$ and

$\rho = \dfrac{1}{M_{x}(s)\, M_{\lambda b}(t)}$ we have

$$L(1-\rho x(s) b(t)) = 0$$

Proof: Claim: $r(x(s)b(t)) = 1$, the spectral radius. We use the lemma that when a & b are free with $\varphi(a) = \varphi(b) = 0$ then

$$r(ab) = \varphi(a^*a)^{1/2}\,\varphi(b^*b)^{1/2}$$

(proof deferred)

Let us show that $\varphi(x(s)^* x(s)) = \varphi(b(t)^* b(t))$
$= M_x(s)\, M_{\lambda b}(t) = \rho$. We'll just do the first.
$x(s) = (1-sx)^{-1} - \varphi((1-sx)^{-1})$ so

$$\varphi(x(s)^* x(s)) = \varphi(((1-sx)(1+sx))^{-1}) - |\varphi((1-sx)^{-1})|^2$$

$$= \varphi((1+s^2x^2)^{-1}) - |M_x(s)|^2$$

$$= \int \frac{1}{1+s^2w^2}\, d\mu_x(w) - M_x(s)^2$$

$$= M_x(s) - M_x(s)^2 = M_x(s)(1 - M_x(s))$$

$$= M_x(s)\, M_{\lambda b}(t).$$

Sublemma If $\varphi(a) = 0$ and $r(a) \leq 1$ then $L(1-a) = 0$.

Proof: If $r(a) < 1$ then we can expand

$$\varphi(\log(1-a)) = -\sum_{n\geq 1} \frac{\varphi(a^n)}{n} = 0. \text{ So}$$

$$L(1-a) = \mathrm{Re}(\varphi(\log(1-a))) = 0.$$

Now suppose $r(a) = 1$. For $0 \leq t < 1$ $L(1-ta) = 0$. Recall that L is upper semi continuous. So $L(1-a) \geq \overline{\lim_{t\to 1^-}} L(1-ta) = 0$

Thus $0 \leq L(1-a)$. Next for $\varepsilon > 0$ by the subharmonicity of L we have

⑥ $L(1-a) \leq \max_{|z| \leq 1+\varepsilon} L(z-a) = \max_{|z|\leq 1+\varepsilon}\{L(z) + L(1-\bar{z}a)\}$

$= \max_{|z|\leq 1+\varepsilon} \log|z| = \log(1+\varepsilon)$. Hence

for all $\varepsilon > 0$ $\quad 0 \leq L(1-a) \leq \log(1+\varepsilon)$.

thus $L(1-a) = 0$.