Hoeffding's Inequality: $a \le X \le b$ ①

$$\Rightarrow E\left(e^{t(X-E(X))}\right) \le \exp\left(\tfrac{1}{8}t^2(b-a)^2\right), \text{ for } t>0.$$

Proof: Since $a \le X \le b$ we know $E(|X|) < \infty$ so $a - E(X) \le X - E(X) \le b - E(X)$

Let $a' = a - E(X)$, $b' = b - E(X)$ then $a - b = a' - b'$. So we assume $E(X) = 0$ and $a \le X \le b$. This implies $a \le 0$ $b \geq 0$.

If $a \le x \le b$ then

$$x = \frac{x-a}{b-a} \cdot b + \frac{b-x}{b-a} \cdot a$$

$$tx = \frac{x-a}{b-a}(tb) + \frac{b-x}{b-a}(ta)$$

$$e^{tx} \le \frac{x-a}{b-a} e^{tb} + \frac{b-x}{b-a} e^{ta}$$

thus $e^{tX} \le \dfrac{X-a}{b-a} e^{tb} + \dfrac{b-X}{b-a} e^{ta}$

$$\boxed{E(e^{tX}) \le \frac{be^{ta} - ae^{tb}}{b-a}}.$$

So now we use calculus to prove

$$\boxed{\frac{be^{ta} - ae^{tb}}{b-a} \le \exp\left(\frac{1}{8}t^2(b-a)^2\right)}$$ for all t.

② it suffices to show

$$\log\left[\frac{be^{ta}-ae^{tb}}{b-a}\right] \leq \frac{[t(b-a)]^2}{8}$$

$$\log\left[\frac{be^{ta}-ae^{tb}}{b-a}\right] = \log\left[e^{ta}\cdot\left(b-ae^{t(b-a)}\right)(b-a)^{-1}\right]$$

$$= ta + \log\left(b-ae^{t(b-a)}\right) - \log(b-a)$$

$$= t(b-a)\frac{a}{b-a} + \log\left(b-ae^{t(b-a)}\right) - \log(b-a)$$

Let $x = t(b-a)$ and

$$f(x) = x\frac{a}{b-a} + \log(b-ae^{x}) - \log(b-a)$$

$$f(0) = 0$$

$$f'(x) = \frac{a}{b-a} - \frac{ae^{x}}{b-ae^{x}}, \quad f'(0) = 0$$

$$f''(x) = \frac{-abe^{x}}{(b-ae^{x})^2} \leq \frac{1}{4}$$

because $-4abe^{x} \leq (b-ae^{x})^2$. By Lagrange's formula for the remainder in Taylor's series there is θ between 0 and x such that

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(\theta x)\frac{x^2}{2!} \leq \frac{x^2}{8}$$

Hence $E(e^{tX}) \le e^{t^2(b-a)^2/8}$

Hoeffding's Inequality, II

$X_1, \dots, X_n$ independent $\quad a_i \le X_i \le b_i$

$K_i = b_i - a_i$, $K = \sqrt{K_1^2 + \cdots + K_n^2}$, $\mathring{X}_i = X_i - E(X_i)$

$$P(\sum_i \mathring{X}_i \ge t) = P(\lambda \sum_i \mathring{X}_i \ge \lambda t) \quad (\text{for } \lambda > 0)$$

$$= P(\exp(\lambda \sum_i \mathring{X}_i) \ge e^{\lambda t})$$

$$\le e^{-\lambda t} E(\exp(\sum_i \lambda \mathring{X}_i))$$

$$= e^{-\lambda t} \prod_i E(e^{\lambda \mathring{X}_i}) = e^{-\lambda t} e^{\frac{t^2}{8}(K_1^2 + \cdots + K_n^2)}$$

$$= e^{-\lambda t} e^{\lambda^2 K^2/8}$$

$$P(-\sum_i \mathring{X}_i \ge t) \le e^{-\lambda t} e^{\lambda^2 K^2/8}$$

$$P(|\sum_{i=1}^n \mathring{X}_i| \ge t) \le e^{-\lambda t} e^{\lambda^2 K^2/8}, \quad \forall \lambda > 0.$$

choose $\lambda > 0$ to minimize $-\lambda t + \frac{\lambda^2 K^3}{8}$:

$\lambda = \frac{4t}{K^2}$. @ $\lambda = \frac{4t}{K^2}$ $\quad -\lambda t + \frac{\lambda^2 K^2}{8} = \frac{-2t^2}{K^2}$

$$\boxed{P(|\sum_{i=1}^n X_i - E(X_i)| \ge t) \le 2e^{-2t^2/K^2}}$$

④ Hoeffding's Inequality, III

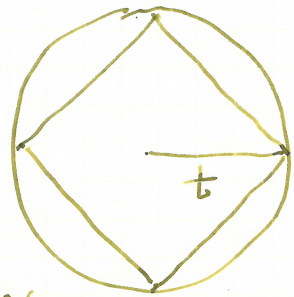
$Z_1, \dots, Z_n$ independent complex valued random variables, $|Z_i| \le M_i$,
$M^2 = M_1^2 + \dots + M_n^2$.

$$\boxed{P\left(\left|\sum_{i=1}^{n} \mathring{Z}_i\right| \ge t\right) \le 4\, e^{-t^2/(8M^2)}.}$$

Proof: $P\left(\left|\sum_{i=1}^{n} \mathring{Z}_i\right| \ge t\right) \le P\left(\left|\sum_{i=1}^{n} \mathrm{Re}(\mathring{Z}_i)\right| + \left|\sum \mathrm{Im}(\mathring{Z}_i)\right| \ge t\right)$

$\le P\left(\left|\sum \mathrm{Re}(\mathring{Z}_i)\right| \ge t/2\right) + P\left(\left|\sum \mathrm{Im}(\mathring{Z}_i)\right| \ge t/2\right)$

$\le 4\, e^{-2t^2/[4\cdot 4\cdot M^2]} = 4\, e^{-t^2/(8M^2)}$



Fubini and Expectation

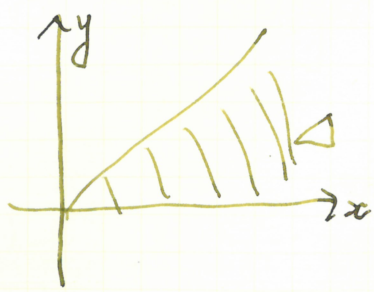
$X \ge 0$ with distribution μ_X, $E(X) < \infty$

$$\Rightarrow E(X) = \int_0^\infty P(X \ge t)\, dt$$

Proof: $E(X) = \int_0^\infty x \, d\mu_X(x) = \int_0^\infty \int_0^\infty \mathbb{1}_\Delta(x,y)\, dy\, d\mu_X(x)$

$= \int_0^\infty \int_0^\infty \mathbb{1}_\Delta(x,y)\, d\mu_X(x)\, dy$

$= \int_0^\infty P(X \ge y)\, dy$



Lemma 2.3.1 (Tao)

Let X be a $N \times N$ random matrix, $X = (x_{ij})_{i,j=1}^{N}$ with $E(x_{ij}) = 0$, $|x_{ij}| \leq 1$ $\forall i, j$. Suppose $\xi \in \mathbb{C}^N$ with $\|\xi\| = 1$, and the random variables $\{x_{ij}\}_{i,j=1}^{N}$ form an independent set. Then for $\lambda > 0$ we have

$$\boxed{P(\|X\xi\| \geq \lambda\sqrt{N}) \leq e^{-N(\lambda^2 - \log 32)}}$$

Remark when $\lambda > 3/2$ $\lambda < \lambda^2 - \log(32)$.

Proof: Step I $P(\|X_i \cdot \xi\| \geq \lambda) \leq 4e^{-\lambda^2/8}$

where $X = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix}$ are the rows of X.

$X_i \cdot \xi = \sum_{j=1}^{N} \xi_j x_{ij}$ $\qquad |\xi_j \cdot x_{ij}| \leq |\xi_j|$ So

by Hoeffding III $P(|X_i \cdot \xi| \geq \lambda) \leq 4e^{-\lambda^2/8}$.

Step II $E(e^{\|X_i \cdot \xi\|^2}) \leq 32$.

$$E(e^{\|X_i \cdot \xi\|^2}) = \int_0^\infty P(\|X_i \cdot \xi\|^2 \geq t)\, dt \quad \left\{\begin{matrix}\text{by}\\ \text{Fubini}\end{matrix}\right\}$$

$$= \int_0^\infty P(\|X_i \cdot \xi\| \geq \sqrt{t})\, dt = \int_0^\infty P(\|X_i \cdot \xi\| \geq s)\, 2s\, ds$$

$$\leq \int_0^\infty 8 e^{-t^2/8} dt = 32 \int_0^\infty d(e^{-s^2/8}) = 32$$

Step III $P(\|X\xi\| \geq \lambda\sqrt{N}) \leq e^{-N(\lambda^2 - 5\log 2)}$

⑥ $P(\|X\xi\| \geq \lambda\sqrt{N}) = P(\|X\xi\|^2 \geq \lambda^2 N)$

$$= P(\exp(\|X\xi\|^2) \geq e^{\lambda^2 N})$$

$$\leq e^{-\lambda^2 N} E(e^{\|X_1\xi\|^2} \cdots e^{\|X_N\xi\|^2})$$

$$= e^{-\lambda^2 N} E(e^{\|X_1\cdot\xi\|^2}) \cdots E(e^{\|X_N\cdot\xi\|^2})$$

$$\leq e^{-\lambda^2 N} 32^N = e^{-N(\lambda^2 - \log 32)}.$$

ε-nets

Definition Let S be the unit sphere of $\mathbb{C}^N$. $\Sigma \subseteq S$ is a maximal ε-net if

(i) $\|\xi_1 - \xi_2\| > \varepsilon$ for $\xi_1, \xi_2 \in \Sigma$, $\xi_1 \neq \xi_2$

(ii) $\forall \eta \in S$ $\exists \xi \in \Sigma$ s.t. $\|\eta - \xi\| \leq \varepsilon$.

Lemma Let X be a random $N \times N$ matrix and $\lambda > 0$ we have for $\Sigma \subseteq S$, a maximal ε-net

$$P(\|X\| > \lambda) \leq P\Big(\bigvee_{\xi \in \Sigma} \big(\|X\xi\| > (1-\varepsilon)\lambda\big)\Big).$$

Remark $\{\omega \mid \|X(\omega)\| > \lambda\}$

$= \{\omega \mid \exists \xi \in S \text{ s.t. } \|X(\omega)\xi\| > \lambda\}$. We shall show

$\{\omega \mid \exists \xi \in S \text{ s.t. } \|X(\omega)\xi\| > \lambda\}$

$\subseteq \{\omega \mid \exists \xi \in \Sigma, \text{ s.t. } \|X(\omega)\xi\| > (1-\varepsilon)\lambda\}.$

Thus we have to show that $\|X(\omega)\| > \lambda$ ⑦

$\Rightarrow \exists\ \xi \in \Sigma$ s.t. $\|X(\omega)\xi\| > (1-\varepsilon)\lambda$. This is an easy application of the triangle inequality.

Proof: Let $\omega \in \Omega$ and suppose $\|X(\omega)\| > \lambda$. Then $\exists\ \xi_0 \in S$ such that $\|X(\omega)\|$

$= \|X(\omega)\xi_0\|$. Then choose $\xi \in \Sigma$ such that $\|\xi - \xi_0\| \leq \varepsilon$. Then $\|X(\omega)\xi - X(\omega)\xi_0\|$

$\leq \varepsilon \|X(\omega)\|$. Hence

$$-\varepsilon\|X(\omega)\| \leq \|X(\omega)\xi\| - \|X(\omega)\xi_0\| = \|X(\omega)\xi\| - \|X(\omega)\|$$

Thus $\|X(\omega)\xi\| \geq (1-\varepsilon)\|X(\omega)\| > (1-\varepsilon)\lambda$.

Lemma Let Σ be an ε-net in $S \subseteq \mathbb{C}^N$. Then

$$|\Sigma| \leq (1+2\varepsilon^{-1})^N = e^{N\log(1+2\varepsilon^{-1})} \quad \text{for } \varepsilon \leq 1/2$$

Proof Let $B_r(\xi) = \{\eta \mid \|\eta - \xi\| \leq r\}$. $\mathrm{vol}(B_r) = C_N r^N$ where $C_N = \dfrac{\pi^{N/2}}{\Gamma(N/2+1)}$. $|\Sigma|\,\mathrm{vol}(B_{\varepsilon}) \leq \mathrm{vol}(B_{1+\varepsilon})$

Thus $|\Sigma| \leq \left(\dfrac{1+\varepsilon/2}{\varepsilon/2}\right)^N = (1+2\varepsilon^{-1})^N$

- each $B_{\varepsilon/2}(\xi_1) \cap B_{\varepsilon/2}(\xi_2) \neq \emptyset$

$\Rightarrow \|\xi_1 - \xi_2\| < \varepsilon$.

8

GUE random matrices

$X \sim \mathcal{N}_{\mathbb{R}}(0,1)$ means X is a normally distributed real random variable with mean 0 and variance 1, $X \sim \mathcal{N}_{\mathbb{C}}(0,1)$ means X is a complex random variable with real and imaginary parts that are independent, normally distributed, centred, with variance $\frac{1}{2}$. for such an X $E(|X|^2) = 1$, $E(X^2) = 0$.

$X_N = \frac{1}{\sqrt{N}} (x_{ij})_{i,j=1}^{N}$ is a GUE random matrix if

- $X_N = X_N^*$
- $\{x_{11}, \dots, x_{NN}, \mathrm{Re}(x_{12}), \dots, \mathrm{Re}(x_{N-1,N}), \mathrm{Im}(x_{12}), \dots, \mathrm{Im}(x_{N-1,N})\}$ are independent and normally distributed such that $x_{ii} \in \mathcal{N}_{\mathbb{R}}(0,1)$, $x_{ij} \in \mathcal{N}_{\mathbb{C}}(0,1)$, $i \neq j$.

$\mathrm{tr} = \frac{1}{N} \mathrm{Tr}$

$$E(\mathrm{tr}(X_N^k)) \to \int_{-2}^{2} t^k \, d\nu_0(t)$$

where $d\nu_0(t) = \frac{\sqrt{4-t^2}}{2\pi} \, dt$ on $[-2,2]$,

Wigner's semi circle law.

Now suppose $X_1, \ldots, X_r$ are independent $N \times N$ GUE random matrices and $p \in \mathbb{C}\langle a_1, \ldots, a_r\rangle$ the polynomials in the variables $a_1 \ldots a_n$, not assuming they commute. We say p is self-adjoint if when we put an involution $*$ on $\mathbb{C}\langle a_1, \ldots, a_r\rangle$ by setting $a_i^* = a_i$ $1 \le i \le r$ and

$(a_{i_1} \cdots a_{i_n})^* = a_{i_n} \cdots a_{i_1}$, we have

$p^* = p$. For example $p = i a_1 a_2 a_3 + a_2 - i a_3 a_2 a_1$.

<u>Theorem 4.1 (Chen, Garza-Vargas, van Handel, II)</u>

Let q be the degee of p. There are constants $C, c > 0$, independent of r, q, N, and a sequence $\{\nu_k\}_{k=0}^{\infty}$ of compactly distributed "distributions" (depending on p) such that for all $h \in C^\infty(\mathbb{R})$ $1 \le m \le N/2$

$$\left| E(\mathrm{tr}(h(p(X_1, \ldots, X_r)))) - \sum_{k=0}^{m-1} \frac{\nu_k(h)}{N^k} \right|$$

$$\leq \frac{(Cq)^{2m}}{m!\, N^m} \| f^{(2m+1)} \|_{[0,2\pi]}$$

$$+ \; C r \, e^{-cN} \left\{ \|h\|_\infty + \| f' \|_{[0,2\pi]} \right\}$$

where $K = (Cr)^q \| p(x_1, \ldots, x_r) \|$ and

$$f(\theta) = h(K \cos \theta).$$

By $x_1, \ldots, x_n \in B(H)$ we mean a free semi-circular family