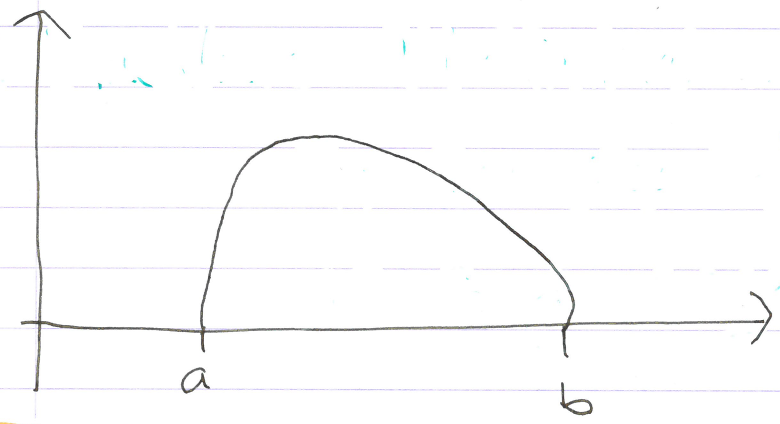

Introduction to Wishart Matrices · 18 Oct. 2022

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Suppose $M, N \geqslant 1$. We shall suppose that M is an implicit function of N.

For $1 \leq i \leq M$, $1 \leq j \leq N$ let $\{g_{ij}\}_{ij}$ be a rectangular array of independent identically distributed complex Gaussian random variables with $E(g_{ij}) = 0$ and $E(|g_{ij}|^2) = 1$. Let $G = (g_{ij})$ be a $M \times N$ matrix and $X_N = \frac{1}{N} G^* G$. X_N is a Wishart matrix

② with shape parameter (M,N). We write $X_N : \Omega \longrightarrow M_N(\mathbb{C})_{s.a.}$ so for each $\omega \in \Omega$ $X_N(\omega)$ has eigenvalues $\lambda_1(\omega) \le \lambda_2(\omega) \le \cdots \le \lambda_N(\omega)$. We let $\mu_N^{(\omega)}$ be the probability measure $\frac{1}{N}(\delta_{\lambda_1(\omega)} + \cdots + \delta_{\lambda_N^{(\omega)}})$ on $\mathbb{R}^+ = [0,\infty)$. We know then $\mathrm{Tr}(X_N^n(\omega)) = \int t^n \, d\mu_N^{(\omega)}(t)$ and $E(\mathrm{Tr}(X_N^n)) = \int t^n \, d\mu_N(t)$ where $\mu_N = \int \mu_N^{(\omega)} \, dP(\omega)$. There are explicit formulas for μ_N using Laguerre polynomials.

Our goal is to find the weak limit of $\{\mu_N\}_{N=1}^{\infty}$, if it exists. For it to exist it suffices to have $\lim_{N\to\infty} \frac{M}{N} = c \in (0,\infty)$. In which case the measures converge to μ_c, the Marchenko-Pastur law. For $c \ge 1$ μ_c has a density $\rho_c(t) = \frac{\sqrt{(b-t)(t-a)}}{2\pi t}$ on the interval $[a,b]$ where $a = (1-\sqrt{c})^2$, $b = (1+\sqrt{c})^2$.

For $0<c<1$ $\int_a^b \rho_c(t)\,dt = c$ and we have in addition an atom at 0 of mass $(1-c)$. So $\mu_c = (1-c)\,\delta_0 + \rho_c(t)\,dt$. We shall prove this by showing that $\lim_{N\to\infty} E(\mathrm{tr}(X_N^n)) = \int t^n\, d\mu_c(t)$, where $\mathrm{tr} = \frac{1}{N}\mathrm{Tr}$.

Computing Traces

$$\mathrm{Tr}((G^*G)^n) = \mathrm{Tr}(G^*G\,G^*G \cdots G^*G)$$

$$= \sum_{i_1,\dots,i_n=1}^{N} \sum_{j_1,\dots,j_n=1}^{M} (G^*)_{i_1 j_1} (G)_{j_1 i_2} \cdots (G^*)_{i_n j_n} (G)_{j_n i_1}$$

$$= \sum_{i_1,\dots,i_n=1}^{N} \sum_{j_1,\dots,j_n=1}^{M} \overline{g}_{j_1 i_1}\, g_{j_1 i_2}\, \overline{g}_{j_2 i_2}\, g_{j_2 i_3} \cdots \overline{g}_{j_n i_n}\, g_{j_n i_1}$$

$$= \sum_{i_1,\dots,i_n=1}^{N} \sum_{j_1,\dots,j_n=1}^{M} g_{j_1 i_2}\, g_{j_2 i_3} \cdots g_{j_n i_1}\, \overline{g}_{j_1 i_1}\, \overline{g}_{j_2 i_2} \cdots \overline{g}_{j_n i_n}$$

Back to Gaussian random variables: $Z_1, \dots Z_s$ independent complex Gaussian random variables with $E(Z_i)=0$, $E(|Z_i|^2)=1$. We saw that $E(Z_{i_1} \cdots Z_{i_n} \overline{Z}_{j_1} \cdots \overline{Z}_{j_n})$ is the number of pairings π of $[2n]$ such that

4

we pair a Z_{i_r} with a $\overline{Z}_{j_s}$. Such a π is described by a permutation $\sigma \in S_n$ as follows. Let $\sigma(r) = s$. Conversely given $\sigma \in S_n$ we let $(r, \sigma(r))$ be a pair of π. Thus we can count the pairings with permutations.

Lemma $E(Z_{i_1} \cdots Z_{i_n} \overline{Z}_{j_1} \cdots \overline{Z}_{j_n}) = |\{\sigma \in S_n \mid i = j \circ \sigma\}|$

Now let us compute $E(g_{j_1 i_2} \cdots g_{j_n i_1} \overline{g}_{j_1 i_1} \cdots \overline{g}_{j_n i_n})$.

Let $\alpha_k = (j_k, i_{k+1})$ with $\alpha_n = (j_n, i_1)$ and $\beta_k = (j_k, i_k)$. Then $E(g_{j_1 i_2} \cdots g_{j_n i_1} \overline{g}_{j_1 i_1} \cdots \overline{g}_{j_n i_n})$
$= E(g_{\alpha_1} \cdots g_{\alpha_n} \overline{g}_{\beta_1} \cdots \overline{g}_{\beta_n}) = |\{\sigma \in S_n \mid i = j \circ \sigma\}|$.

If $i = j \circ \sigma$ then $(j_k, i_{k+1}) = \alpha_k = \beta_{\sigma(k)}$
$= (j_{\sigma(k)}, i_{\sigma(k)})$ so $j_k = j_{\sigma(k)}$. Considering $j : [n] \longrightarrow [M]$ we have $j = j \circ \sigma$ i.e. j is constant on the cycles of σ. The other condition $i_{k+1} = i_{\sigma(k)}$ can be written as follows. Let $\gamma = (1, 2, \ldots, n) \in S_n$. Then our

condition becomes $i \circ \gamma = i \circ \sigma$ or

$i = i \circ \gamma\sigma^{-1}$, thus i is constant on the cycles of $\gamma\sigma^{-1}$. We let for any permutation σ, $\#(\sigma)$ be the number of cycles in the cycle decomposition of σ.

Review of Cycle Counting

A Transposition is a permutation with only one cycle and this cycle has 2 elements. Every permutation can be written as a product of transpositions but not uniquely. If $(i_1, \ldots, i_k)$ is a cycle of σ we can write $(i_1, i_2, \ldots, i_k) = (i_1, i_2)(i_2, i_3) \cdots (i_{k-1}, i_k)$ as a product of $k-1$ transpositions. If $\sigma \in S_n$ has l cycles then we can write σ is a product of $n - \#(\sigma)$ transpositions. In fact this is minimal. We define $|\sigma| = \min\{k \mid \sigma \text{ is a product of } k \text{ transposions}\}$

Lemma Let $\sigma \in S_n$ and τ be a transposition $\tau = (r, s)$.

(6) (a) Then $\#(\sigma\tau) = \begin{cases} \#(\sigma)+1 & r \,\&\, s \text{ are in different cycles of } \sigma \\ \#(\sigma) & r \,\&\, s \text{ are in the same cycles of } \sigma. \end{cases}$

(b) $\#(\sigma) + |\sigma| = n$.

Proof: $(j_1, \dots, j_s)(j_{s+1}, \dots, j_t)(j_s, j_t)$

$= (j_1, \dots, j_t)$. This proves (a). Suppose

$\sigma = \tau_1 \cdots \tau_k$ is a product of transpositions.

Then by (a) $\#(\sigma) \geq \#(\sigma\tau_k) - 1 \geq \#(\sigma\tau_k\tau_{k-1}) - 2$

$\geq \cdots \geq \#(\sigma\tau_k\tau_{k-1}\cdots\tau_1) - k = n - k$ because

$\sigma\tau_k \cdots \tau_1 = e$ and $\#(e) = n$. This shows

that for every factorization $k \geq n - \#(\sigma)$.

But we had previously showed that

we can find a factorization with $n - \#(\sigma)$

factors; hence this is mimal so

$|\sigma| = n - \#(\sigma)$.

Triangle Inequality: $|\sigma_1\sigma_2| \leq |\sigma_1| + |\sigma_2|$

So we have reached the stage

$$E(\mathrm{tr}(X_N^n)) = N^{-(n+1)} \sum_{i_1,\dots,i_n=1}^{N} \sum_{j_1,\dots,j_n=1}^{M} |\{\sigma \in S_n \mid i = i\circ\gamma\sigma^{-1}, j = j\circ\sigma\}|$$

$$= N^{-(n+1)} \sum_{i_1,\dots,i_n=1}^{N} \sum_{j_1,\dots,j_n=1}^{M} \sum_{\sigma\in S_n} \delta_{i,i\circ} \; \delta_{j,j\circ\sigma}$$

$$= \sum_{\sigma \in S_n} N^{-(n+1)} M^{\#(\sigma)} N^{\#(\sigma^{-1}\gamma)}$$

$$\boxed{= \sum_{\sigma \in S_n} \left(\frac{M}{N}\right)^{\#(\sigma)} N^{\#(\sigma)+\#(\sigma^{-1}\gamma)-(n+1)} .}$$

Analysis of $\#(\sigma)+\#(\sigma^{-1}\gamma)-(n+1)$

Let $NC(n) = \{\sigma \in S_n \mid \#(\sigma)+\#(\sigma^{-1}\gamma)=n+1\}$.

For example $e, \gamma \in NC(n)$. $\#(\sigma)+\#(\sigma^{-1}\gamma)$

$= 2n + \#(\sigma)-n + \#(\sigma^{-1}\gamma)-n = 2n-(|\sigma|+|\sigma^{-1}\gamma|)$

$\geqslant 2n-(|\gamma|) = n+1$. Hence

$\#(\sigma)+\#(\sigma^{-1}\gamma)-(n+1) \leq 0$ with equality only if $\sigma \in NC(n)$. Thus

$$\lim_{N\to\infty} E(tr(X_N^n)) = \sum_{\sigma \in NC(n)} c^{\#(\sigma)}$$

where $c = \lim_{N\to\infty} \frac{M}{N}$ (which we have to assume).

Lemma (a) $NC(n) = \{\sigma \mid |\sigma|+|\sigma^{-1}\gamma| = |\gamma|\}$.

(b) $NC(n) = \{\sigma \mid \exists\ \tau_1, \ldots, \tau_{n-1}$ transpositions such that $\gamma = \tau_1 \cdots \tau_{n-1}$ and $\sigma = \tau_1 \cdots \tau_k$ where $k = |\sigma|\}$

Proof: $n+1 = \#(\sigma)+\#(\sigma^{-1}\gamma) = 2n+\#(\sigma)-n+\#(\sigma^{-1}\gamma)-n = 2n-(|\sigma|+|\sigma^{-1}\gamma|)$

⑧ $\Leftrightarrow$ $|\sigma| + |\sigma^{-1}\gamma| = n-1 = |\gamma|$. This proves (a).

Let $k = |\sigma|$ and write $\sigma = \tau_1 \cdots \tau_k$.

If $|\sigma| + |\sigma^{-1}\gamma| = |\gamma|$ then $|\sigma^{-1}\gamma| = n-1-k$

so we can write $\sigma^{-1}\gamma = \tau_{k+1} \cdots \tau_{n-1}$. Hence

$\gamma = \tau_1 \cdots \tau_{n-1}$. Conversely if $\gamma = \tau_1 \cdots \tau_{n-1}$

and $\sigma = \tau_1 \cdots \tau_k$ then $|\sigma| \leq k$ and $|\sigma^{-1}\gamma|$

$\leq n-1-k$ so $|\sigma| + |\sigma^{-1}\gamma| \leq n-1 = |\gamma| \leq |\sigma| + |\sigma^{-1}\gamma|$.

Hence $|\sigma| + |\sigma^{-1}\gamma| = |\gamma|$ and $\sigma \in NC(n)$. This

proves (b).

Theorem $NC(n)$ is the set of non-crossing

partitions on $[n]$.

Proof: Write $\gamma = \tau_1 \cdots \tau_{n-1}$ with $\sigma = \tau_1 \cdots \tau_k$

$= \gamma \tau_{n-1} \cdots \tau_{k+1}$. So σ is obtained from

γ by multiplying by a transposition which

splits a cycle. Recall that

$(j_1, \ldots, j_r, j_{r+1}, \ldots, j_s)(j_r, j_s)$

$= (j_1 \ldots j_r)(j_{r+1}, \ldots, j_s)$ so each time we split a cycle of $\gamma \tau_{n-1} \cdots \tau_k$ we split a cycle in cyclic order into cycles of cyclic order. Recall that a string is in cyclic order if $(j_1 \ldots j_s)$ is of the form $j_1, \gamma^{l_1}(j_1) \ldots \gamma^{l_{s-1}}(j_1)$ with $1 \le l_1 \le \cdots \le l_{s-1} \le n-1$. Thus if $i < j < k < l$ are in cyclic order we cannot have i & k in one cycle and j & l in another cycle. Hence σ is non-crossing.