$$= \sum_{\sigma \in S_n} N^{-(n+1)} M^{\#(\sigma)} N^{\#(\sigma^{-1}\gamma)}$$

$$= \boxed{\sum_{\sigma \in S_n} \left(\frac{M}{N}\right)^{\#(\sigma)} N^{\#(\sigma)+\#(\sigma^{-1}\gamma)-(n+1)}} .$$

Analysis of $\#(\sigma)+\#(\sigma^{-1}\gamma)-(n+1)$

25.OCT.22

Let $NC(n) = \{\sigma \in S_n \mid \#(\sigma) + \#(\sigma^{-1}\gamma) = n+1\}$.

For example $e, \gamma \in NC(n)$. $\#(\sigma) + \#(\sigma^{-1}\gamma)$

$= 2n + \#(\sigma) - n + \#(\sigma^{-1}\gamma) - n = 2n - (|\sigma| + |\sigma^{-1}\gamma|)$

$\geq 2n - (|\gamma|) = n+1$. Hence

$\#(\sigma) + \#(\sigma^{-1}\gamma) - (n+1) = 0$ with equality

only if $\sigma \in NC(n)$. Thus

$$\lim_{N\to\infty} E(\mathrm{tr}(X_N^n)) = \sum_{\sigma \in NC(n)} c^{\#(\sigma)}$$

where $c = \lim_{N\to\infty} \frac{M}{N}$ (which we have to assume).

Lemma (a) $NC(n) = \{\sigma \mid |\sigma| + |\sigma^{-1}\gamma| = |\gamma|\}$.

(b) $NC(n) = \{\sigma \mid \exists\ \tau_1, \ldots, \tau_{n-1}$ transpositions such that

$\gamma = \tau_1 \cdots \tau_{n-1}$ and $\sigma = \tau_1 \cdots \tau_k$ where $k = |\sigma|\}$

Proof: $n+1 = \#(\sigma) + \#(\sigma^{-1}\gamma) = 2n + \#(\sigma) - n + \#(\sigma^{-1}\gamma) - n = 2n - (|\sigma| + |\sigma^{-1}\gamma|)$

(8) $\iff$ $|\sigma| + |\sigma^{-1}\gamma| = n-1 = |\gamma|$. This proves (a).

Let $k = |\sigma|$ and write $\sigma = \tau_1 \cdots \tau_k$.

If $|\sigma| + |\sigma^{-1}\gamma| = |\gamma|$ then $|\sigma^{-1}\gamma| = n-1-k$

so we can write $\sigma^{-1}\gamma = \tau_{k+1} \cdots \tau_{n-1}$. Hence

$\gamma = \tau_1 \cdots \tau_{n-1}$. Conversely if $\gamma = \tau_1 \cdots \tau_{n-1}$

and $\sigma = \tau_1 \cdots \tau_k$ then $|\sigma| \leq k$ and $|\sigma^{-1}\gamma|$

$\leq n-1-k$ so $|\sigma| + |\sigma^{-1}\gamma| \leq n-1 = |\gamma| \leq |\sigma| + |\sigma^{-1}\gamma|$.

Hence $|\sigma| + |\sigma^{-1}\gamma| = |\gamma|$ and $\sigma \in NC(n)$. This proves (b).

Theorem $NC(n)$ is the set of non-crossing partitions on $[n]$.

Proof: Write $\gamma = \tau_1 \cdots \tau_{n-1}$ with $\sigma = \tau_1 \cdots \tau_k$

$= \gamma \tau_{n-1} \cdots \tau_{k+1}$. So σ is obtained from

γ by multiplying by a transposition which splits a cycle. Recall that

$$(j_1, \ldots, j_r, j_{r+1}, \ldots, j_s)(j_r, j_s)$$

$= (j_1 \ldots j_r)(j_{r+1}, \ldots, j_s)$ so each time we split a cycle of $\gamma\tau_{n-1}\cdots\tau_k$ we split a cycle in cyclic order into cycles in cyclic order. Recall that a string is in cyclic order if $(j_1 \ldots j_s)$ is of the form $j_1, \gamma^{l_1}(j_1), \ldots \gamma^{l_{s-1}}(j_1)$ with $1 \leq l_1 \leq \cdots \leq l_{s-1} \leq n-1$. Thus if $i < j < k < l$ are in cyclic order we cannot have i & k in one cycle and j & l in another cycle. Hence σ is non-crossing.

Theorem $\lim_{N\to\infty} E(\mathrm{tr}(X_N^n)) = \sum_{\pi \in NC(n)} c^{\#(\pi)}$

where $c = \lim_{N\to\infty} \frac{M}{N}$.

Poisson Random Variables and their cumulants

We saw that for a standard normal random variable $k_n = \begin{Bmatrix} 0 & n \neq 2 \\ 1 & n = 2 \end{Bmatrix}$.

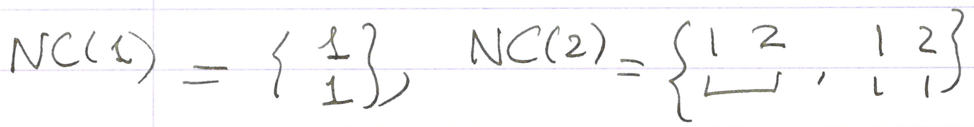
(10) Recall that a random variable is a Poisson random variable if $P(X=k) = e^{-\lambda} \dfrac{\lambda^k}{k!}$. Then $E(e^{itX}) = e^{\lambda(e^{it}-1)}$. So

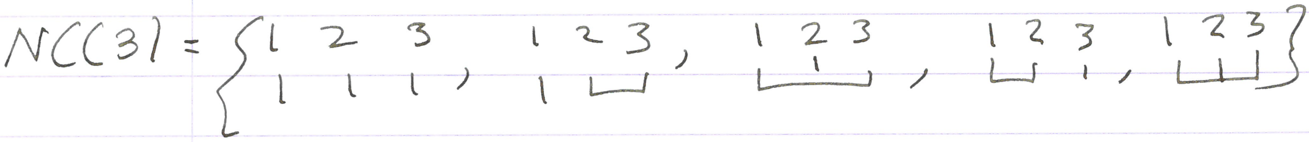
$$\sum_{n=1}^{\infty} k_n \frac{(it)^n}{n!} = \log(E(e^{itX})) = \lambda(e^{it}-1) = \sum_{n=1}^{\infty} \lambda \frac{(it)^n}{n!}.$$

Hence $k_n = \lambda$ for all n. Thus $m_n = \{n^{th} \text{ moment of } X\} = \sum_{\pi \in NC(n)} \lambda^{\#(\pi)}$.

For this reason the Marchenko-Pastur Law is also called the free Poisson law with parameter c. We have the moments of MP_c and we would to find the density.

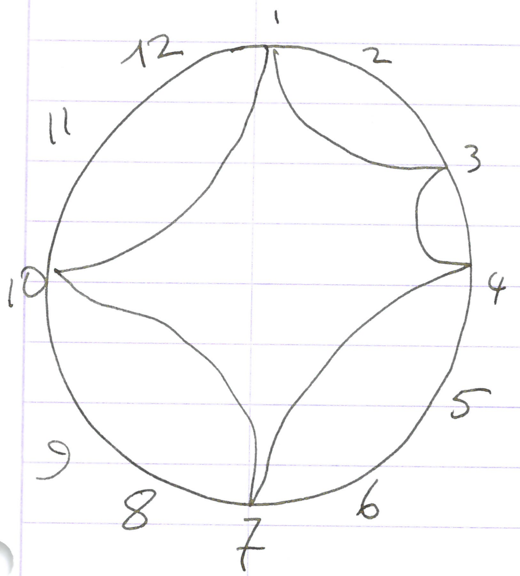
Non-Crossing Partitions





$|NC(n)| = \frac{1}{n+1}\binom{2n}{n}$ so $|NC(4)| = 14$.

Parametrizing Non-Crossing Partitions



Let V be the block containing 1. Let $s = |V|$. Let $i_1, \dots, i_s \geq 0$ be the gaps between adjacent elements

$i_1 + \cdots + i_s - s = n$

In the example $s=5,\ i_1=1,\ i_2=0,\ i_3=2,\ i_4=2,\ i_5=2$

Let $M(z) = 1 + \sum_{n=1}^{\infty} m_n z^n$ be the Moment Generating function and $C(z) = 1 + \sum_{n=1}^{\infty} \kappa_n z^n$ be the cumulant generating function where

$$m_n = \sum_{\pi \in NC(n)} \kappa_\pi$$

is the free moment-cumulant relation.

$m_1 = \kappa_1$

$m_2 = \kappa_2 + \kappa_1^2$

$m_3 = \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3$

$m_4 = \kappa_4 + 4\kappa_1\kappa_3 + 2\kappa_2^2 + 6\kappa_1^2\kappa_2 + \kappa_1^4$

recall that

$m_4 = k_4 + 4k_1k_3 + 3k_2^2 + 6k_1k_2 + k_1^4$

so $k_1 = \kappa_1,\ k_2 = \kappa_2,\ k_3 = \kappa_3$, but $\kappa_4 = k_4 + k_2^2$

(12) Let find the relation between M & C.

$$m_n = \sum_{\pi \in NC(n)} K_\pi = \sum_{s=1}^{n} \sum_{\substack{\pi \in NC(n) \\ 1 \text{ is in a block} \\ \text{of size } s}} K_\pi$$

$$= \sum_{s=1}^{n} K_s \sum_{\substack{i_1, \dots, i_s \geq 0 \\ i_1 + \dots + i_s + s = n}} \sum_{\pi_1 \in NC(i_1)} \sum_{\pi_2 \in NC(i_2)} \dots \sum_{\pi_s \in NC(i_s)} K_{\pi_1} \cdots K_{\pi_s}$$

$$= \sum_{s=1}^{n} K_s \sum_{\substack{i_1, \dots, i_s \geq 0 \\ i_1 + \dots + i_s + s = n}} \left[\sum_{\pi \in NC(i_1)} K_\pi \right] \cdots \left[\sum_{\pi \in NC(i_s)} K_\pi \right]$$

$$= \sum_{s=1}^{n} K_s \sum_{\substack{i_1, \dots, i_s \geq 0 \\ i_1 + \dots + i_s + s = n}} m_{i_1} \cdots m_{i_s}$$

Hence $$m_n z^n = \sum_{s=1}^{n} K_s z^s \sum_{\substack{i_1, \dots, i_s \geq 0 \\ i_1 + \dots + i_s + s = n}} m_{i_1} z^{i_1} \dots m_{i_s} z^{i_s}$$

Thus $$\sum_{n=1}^{\infty} m_n z^n = \sum_{n=1}^{\infty} \sum_{s=1}^{n} K_s z^s \sum_{\substack{i_1, \dots, i_s \geq 0 \\ i_1 + \dots + i_s + s = n}} m_{i_1} z^{i_1} \dots m_{i_s} z^{i_s}$$

$$= \sum_{s=1}^{\infty} K_s z^s \sum_{i_1, \dots, i_s \geq 0} m_{i_1} z^{i_1} \dots m_{i_s} z^{i_s}$$

$$= \sum_{s=1}^{\infty} K_s z^s \sum_{i_1=0}^{\infty} m_{i_1} z^{i_1} \dots \sum_{i_s=0}^{\infty} m_{i_s} z^{i_s}$$

$$= \sum_{s=1}^{\infty} K_s z^s M(z)^s = \sum_{s=1}^{\infty} K_s (z M(z))^s.$$

Hence $M(z) = 1 + \sum_{s \geq 1}^{\infty} m_s z^s = 1 + \sum_{s=1}^{\infty} K_s (z M(z))^s = C(z M(z))$

Theorem Suppose we have two sequences $\{m_n\}_{n=1}^{\infty}$ and $\{\kappa_n\}_{n=1}^{\infty}$. Let $M(z) = 1 + \sum_{n=1}^{\infty} m_n z^n$ and $C(z) = 1 + \sum_{n=1}^{\infty} \kappa_n z^n$. If $m_n = \sum_{\pi \in NC(n)} \kappa_\pi$ then $M(z) = C(zM(z))$.

The Cumulant Generating Function of MP_c

We have $\kappa_n = c$ for $n = 1, 2, 3, \ldots$ Thus

$$C(z) = 1 + cz + cz^2 + \cdots = 1 + \frac{cz}{1-z} = \frac{1-(1-c)z}{1-z}.$$

Hence $M(z) = C(zM(z)) = \dfrac{1-(1-c)zM(z)}{1-zM(z)}$, this gives the quadratic equation in M:

$$zM(z)^2 - (1+(1-c)z)M(z) + 1 = 0$$

$$M(z) = \frac{1+(1-c)z \pm \sqrt{(1+(1-c)z)^2 - 4z}}{2z}$$

Note that $(1+(1-c)z)^2 - 4z = (az-1)(bz-1)$ with $a = (1-\sqrt{c})^2$, $b = (1+\sqrt{c})^2$; $ab = 1-c$, $a+b = 2(1+c)$

Thus $M(z) = \dfrac{1+(1-c)z \pm \sqrt{(az-1)(bz-1)}}{2z}$

$$= \frac{1-c}{2} + \frac{1 \pm \sqrt{(az-1)(bz-1)}}{2z} = \frac{1-c}{2} + \frac{1-\sqrt{(az-1)(bz-1)}}{2z}$$

(14) where we chose "$-$" so that $\lim_{z \to 0} M(z) = 1$.

Hence $\boxed{M(z) = \dfrac{1 + (1-c)z - \sqrt{(az-1)(bz-1)}}{2z}}$.

The Cauchy Transform and Stieltjes Inversion

(How to recover a measure from its moment generating function)