where we chose "$-$" so that $\lim_{z\to 0} M(z) = 1$.

Hence $$\boxed{M(z) = \frac{1+(1-c)z - \sqrt{(az-1)(bz-1)}}{2z}}.$$

The Cauchy Transform and Stieltjes Inversion

(How to recover a measure from its moment generating function)

Lemma Suppose $C(zM(z)) = M(z)$, then $M\left(\frac{z}{C(z)}\right) = C(z)$.

Let $G(z) = z^{-1}M(z^{-1})$ and $K(z) = z^{-1}C(z)$. Then

$$K(G(z)) = z \quad \text{and} \quad G(K(z)) = z$$

Proof: Let $w = zM(z)$. Then $C(w) = C(zM(z)) = M(z)$.

Also $\frac{zM(z)}{C(zM(z))} = \frac{zM(z)}{M(z)} = z$. Thus $C(w) = M(z)$

$$= M\left(\frac{zM(z)}{C(zM(z))}\right) = M\left(\frac{w}{C(w)}\right). \text{ Hence } M\left(\frac{z}{C(z)}\right) = C(z).$$

$$K(G(z)) = \frac{C(G(z))}{G(z)} = \frac{C(z^{-1}M(z^{-1}))}{z^{-1}M(z^{-1})} = \frac{M(z^{-1})}{z^{-1}M(z^{-1})} = z.$$

$$G(K(z)) = \frac{1}{K(z)} M\left(\frac{1}{K(z)}\right) = \frac{z}{C(z)} M\left(\frac{z}{C(z)}\right) = \frac{z\, C(z)}{C(z)} = z.$$

Steiltjes Inversion

8 Nov. 2022

Now let μ be a probability measure on $\mathbb{R}$. Let $\mathbb{C}^+ = \{z \mid \mathrm{Im}(z) > 0\}$, $\mathbb{C}^- = \{z \mid \mathrm{Im}(z) < 0\}$.

For $z \in \mathbb{C}^+$, $t \in \mathbb{R}$ $\quad |z-t|^{-1} \leq [\mathrm{Im}(z)]^{-1}$ Thus $t \mapsto (z-t)^{-1}$ is bounded on $\mathbb{R}$ for $z \in \mathbb{C}^+$.

We let $G(z) = \int_{\mathbb{R}} (z-t)^{-1}\, d\mu(t)$. $\quad G(\mathbb{C}^+) \subseteq \mathbb{C}^-$.

For $\varepsilon > 0$ $\quad \frac{-1}{\pi}\mathrm{Im}(G(x+i\varepsilon)) = \frac{-1}{\pi}\mathrm{Im}\left[\int (x-t+i\varepsilon)^{-1} d\mu(t)\right]$

$$= \frac{-1}{\pi}\int \mathrm{Im}\left[\frac{x-t-i\varepsilon}{(x-t)^2+\varepsilon^2}\right] d\mu(t) = \int \frac{1}{\pi}\frac{\varepsilon}{(x-t)^2+\varepsilon^2}\, d\mu(t)$$

$$= \int \frac{1}{\pi}\frac{1}{1+\left(\frac{x-t}{\varepsilon}\right)^2}\frac{1}{\varepsilon}\, d\mu(t)$$

. For $a \leq b$ we have $\int_a^b \frac{-1}{\pi}\mathrm{Im}(G(x+i\varepsilon))\, dx$

$$= \frac{1}{\pi}\int_a^b \int_{\mathbb{R}} \frac{1}{1+\left(\frac{x-t}{\varepsilon}\right)^2}\frac{1}{\varepsilon}\, d\mu(t)\, dx$$

$$= \int_{\mathbb{R}} \frac{1}{\pi}\int_a^b \frac{1}{1+\left(\frac{x-t}{\varepsilon}\right)^2}\, d\left(\frac{x-t}{\varepsilon}\right)\, d\mu(t)$$

$$= \int_{\mathbb{R}} \frac{1}{\pi}\int_{\frac{a-t}{\varepsilon}}^{\frac{b-t}{\varepsilon}} \frac{1}{1+u^2}\, du\, d\mu(t) = \int_{\mathbb{R}} \frac{1}{\pi}\left(\tan^{-1}\left[\frac{b-t}{\varepsilon}\right] - \tan^{-1}\left(\frac{a-t}{\varepsilon}\right)\right) d\mu(t)$$

$$\lim_{\varepsilon\to 0^+}\int_a^b \frac{-1}{\pi}\mathrm{Im}(G(x+i\varepsilon))\, dx = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} \frac{1}{\pi}\left[\tan^{-1}\left(\frac{b-t}{\varepsilon}\right) - \tan^{-1}\left(\frac{a-t}{\varepsilon}\right)\right] d\mu(t)$$

16

$$\lim_{\varepsilon\to 0^+} \frac{1}{\pi}\left\{\tan^{-1}\left(\frac{b-t}{\varepsilon}\right)-\tan^{-1}\left(\frac{a-t}{\varepsilon}\right)\right\} = \begin{cases} 0 & t<a \text{ or } b<t \\ \frac{1}{2} & t\in\{a,b\} \\ 1 & a<t<b \end{cases}$$

$$= \chi_{[a,b]}(t) - \frac{1}{2}\chi_{\{a,b\}}(t)$$

Thus $\displaystyle\lim_{\varepsilon\to 0^+} \int_a^b \frac{1}{\pi} \operatorname{Im}(G(x+i\varepsilon))\, dx$

$$= \int \chi_{[a,b]}(t) - \frac{1}{2}\chi_{\{a,b\}}(t)\, d\mu(t)$$

$$= \mu([a,b]) - \frac{1}{2}\mu(\{a,b\})$$

Stirling's formula

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

$$(2n)! \sim \left(\frac{2n}{e}\right)^{2n} 2\sqrt{\pi n} = 4^n \left(\frac{n}{e}\right)^{2n} 2\sqrt{\pi n}$$

$$\frac{(2n)!}{(n!)^2} \sim 4^n \left(\frac{n}{e}\right)^{2n} 2\sqrt{\pi n}\left(\frac{e}{n}\right)^{2n}(2\pi n)^{-1}$$

$$= 4^n \frac{1}{\sqrt{\pi n}}$$

$$C_n = \frac{1}{n+1}\binom{2n}{n} \sim 4^n \frac{1}{\sqrt{\pi n}\,(n+1)}$$

$$m_n = \sum_{\pi \in NC(n)} c^{\#(\pi)} \quad , m_n \leq \frac{c^n 4^n}{\sqrt{\pi n}\,(n+1)} + o\left(\frac{1}{n}\right)$$

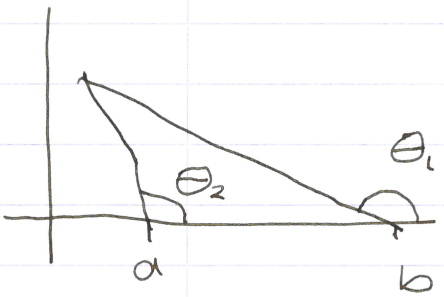
$\limsup\limits_{n} m_n^{1/n} \leq 4c$. So for

$|z| < \frac{1}{4c}$ $\quad M(z) = \sum\limits_{n=0}^{\infty} m_n z^n$ converges.

Likewise for $|z| > 4c$ $\quad G(z) = \frac{1}{z} M\left(\frac{1}{z}\right)$

converges and

$$G(z) = \frac{z + 1 - c - \sqrt{(z-a)(z-b)}}{2z}$$



$$0 \leq \theta_1, \theta_2 \leq \pi$$

$$\sqrt{(z-a)(z-b)} = |(z-a)(z-b)|^{1/2} e^{i(\theta_1+\theta_2)/2}$$

Let $z = x + i\varepsilon$. $\lim\limits_{\varepsilon \to 0^+} \frac{\theta_1 + \theta_2}{2} = \begin{cases} \pi & x < a \\ 3\pi/4 & x = a \\ \pi/2 & a < x < b \\ \pi/4 & x = b \\ 0 & b < x \end{cases}$

$$\lim_{\varepsilon \to 0^+} \sqrt{(z-a)(z-b)} = |(z-a)(z-b)|^{1/2} \times \begin{cases} -1 & x < a \\ i & a < x < b \\ 1 & b < x \end{cases}$$

$$\frac{-1}{\pi} \operatorname{Im}\left[G(x+i\varepsilon)\right] = \frac{-1}{\pi} \Big\{ \operatorname{Im}\left(\frac{1}{2}\right) + (1-c)\operatorname{Im}\left(\frac{1}{z}\right)$$

$$- \frac{|(z-a)(z-b)|^{1/2}}{2} \operatorname{Im}\left[\frac{e^{i\theta}}{x+i\varepsilon}\right] \Big\} \qquad \theta = \frac{\theta_1 + \theta_2}{2}$$

$$\lim_{\varepsilon \to 0^+} \frac{-1}{\pi} \operatorname{Im}\left[G(x+i\varepsilon)\right] = \frac{|(x-a)(x-b)|^{1/2}}{2\pi} \lim_{\varepsilon \to 0} \operatorname{Im}\left[\frac{(x-i\varepsilon)\, e^{i\theta}}{x^2 + \varepsilon^2}\right]$$

18

$$= \frac{|(x-a)(x-b)|^{1/2}}{2\pi x^2} \left[\lim_{\varepsilon \to 0^+} x\sin(\theta) - \varepsilon\cos(\theta)\right]$$

$$= \frac{|(x-a)(x-b)|^{1/2}}{2\pi x} \lim_{\varepsilon \to 0^+} \sin\theta = \frac{|(x-a)(x-b)|^{1/2}}{2\pi} \begin{cases} 0 & x<a \text{ or } x>b \\ 1 & a<x<b. \end{cases}$$

$$= \frac{\sqrt{(b-x)(x-a)}}{2\pi x} \begin{cases} 0 & x<a \text{ or } x>b \\ 1 & a \leq x \leq b \end{cases}$$

$$= \frac{\sqrt{(b-x)(x-a)}}{2\pi x} \mathbb{1}_{[a,b]}(x).$$

$R = (b-x)(x-a)$

$$\int_a^b \frac{\sqrt{R}}{x}\,dx = \int_a^b \frac{d}{dx}\sqrt{R}\,dx + \frac{1}{2}(a+b)\int_a^b \frac{dx}{\sqrt{R}} - ab\int_a^b \frac{dx}{x\sqrt{R}}$$

$$\int_a^b \frac{d}{dx}\sqrt{R}\,dx = 0, \quad \frac{a+b}{2}\int_a^b \frac{dx}{\sqrt{R}} = \frac{a+b}{2}\cdot\pi$$

$$\int_a^b \frac{-ab}{x\sqrt{R}}\,dx = -\pi\sqrt{ab}.$$ Thus

$$\int_a^b \frac{\sqrt{R}}{2\pi x}\,dx = \frac{a+b}{2} - \sqrt{ab} = \left(\frac{\sqrt{b}-\sqrt{a}}{2}\right)^2 = \begin{cases} c & c \leq 1 \\ 1 & c > 1. \end{cases}$$

So $\frac{\sqrt{(b-x)(x-a)}}{2\pi x} \mathbb{1}_{[a,b]}(x)$ is a probability measure for $c \geq 1$, and a sub robability measure for $c < 1$.

Back to Marchenko–Pastur

$$G(z) = \frac{z+1-c-\sqrt{(z-a)(z-b)}}{2z} = \frac{f(z)}{z}$$

where $f(z) = \dfrac{z+1-c-\sqrt{(z-a)(z-b)}}{2}$. So $\lim\limits_{\angle z\to 0} z\,G(z)$

$$= \lim_{\angle z\to 0} f(z) = \frac{1}{2}\left(1-c - \lim_{\angle z\to 0}\sqrt{(z-a)(z-b)}\right)$$

$$= \frac{1}{2}\left(1-c - \lim_{\angle z\to 0}|(z-a)(z-b)|^{1/2}\, e^{i\theta}\right)$$

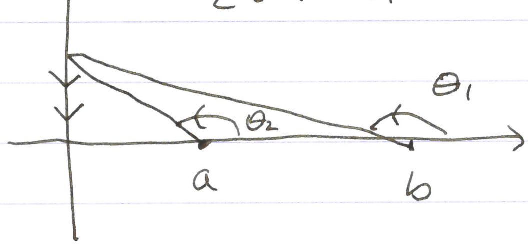
$$= \frac{1}{2}\left(1-c - \sqrt{ab}\lim_{\angle z\to 0} e^{i\theta}\right)$$

$$\left\{\begin{array}{l} a = (1-\sqrt{c})^2 \\ b = (1+\sqrt{c})^2 \\ ab = (1-c)^2 \end{array}\right.$$

$$= \frac{1}{2}\left(1-c - |1-c|\lim_{\angle z\to 0} e^{i\theta}\right)$$

$$\sqrt{ab} = |1-c|$$

$$\theta = \tfrac{1}{2}(\theta_1+\theta_2) \to \pi$$



$$= \frac{1-c+|1-c|}{2}$$

$$= \begin{cases} 1-c & 0<c<1 \\ 0 & c>1 \end{cases}$$

$$\boxed{\text{Thus}\quad d\nu_c(t) = \begin{cases} (1-c)\delta_0 + \dfrac{\sqrt{(b-t)(t-a)}}{2\pi t} & 0<c<1 \\[2ex] \dfrac{\sqrt{(b-t)(t-a)}}{2\pi t} & c\geq 1 \end{cases}}$$

Integration of $\frac{\sqrt{(b-x)(x-a)}}{2\pi x}$ over $[a,b]$

$$\begin{aligned} a &= (1-\sqrt{c})^2 = 1+c-2\sqrt{c} \\ b &= (1+\sqrt{c})^2 = 1+c+2\sqrt{c} \\ a+b &= 2(1+c) \\ ab &= (1-c)^2 = (1+c)^2-4c \end{aligned}$$

$$R = (b-x)(x-a) = -x^2+(a+b)x-ab = -x^2+2(1+c)x-(1+c)^2+4c$$

$$\frac{\sqrt{R}}{x} = \frac{R}{x\sqrt{R}} = \frac{-x+(a+b)}{\sqrt{R}} - \frac{ab}{x\sqrt{R}} = \frac{1}{2}\,\frac{-2x+a+b}{\sqrt{R}} + \frac{1}{2}\,\frac{a+b}{\sqrt{R}} - \frac{ab}{x\sqrt{R}}$$

$$= \underbrace{\frac{d}{dx}\sqrt{R}}_{\boxed{1}} + \underbrace{\frac{1}{2}\,\frac{a+b}{\sqrt{R}}}_{\boxed{2}} - \underbrace{\frac{ab}{x\sqrt{R}}}_{\boxed{3}}.$$

$\boxed{1}$ $\displaystyle\int_a^b \frac{d}{dx}\sqrt{R}\,dx = \sqrt{R(b)} - \sqrt{R(a)} = 0-0 = \boxed{0}$

$\boxed{2}$ $R = 4c-(x-(1+c))^2 = c\left(4-\left(\frac{x-(1+c)}{\sqrt{c}}\right)^2\right)$. $\sqrt{R} = \sqrt{c}\sqrt{4-t^2}$ where

$t = \frac{x-(1+c)}{\sqrt{c}}$ $\quad dt = \frac{dx}{\sqrt{c}}$. Thus $\frac{dx}{\sqrt{R}} = \frac{dx}{\sqrt{c}\sqrt{4-t^2}} = \frac{dt}{\sqrt{4-t^2}}$. Hence

$$\int_a^b \frac{a+b}{2}\,\frac{dx}{\sqrt{R}} = \frac{a+b}{2}\int_{-2}^{2}\frac{dt}{\sqrt{4-t^2}} = \frac{a+b}{2}\int_0^{\pi} d\theta = \boxed{\frac{a+b}{2}\cdot\pi}$$

$t = 2\cos\theta$, $dt = -2\sin\theta\,d\theta$
@ $t=-2$, $\theta=\pi$
@ $t=2$, $\theta=0$

$\boxed{3}$ $R = -x^2+(a+b)x-ab$

$$= ab\left(-1+\frac{a+b}{ab}x-\frac{x^2}{ab}\right)$$

$u = \frac{b-a}{2ab}$, $\quad v = \frac{b+a}{2ab}$, $\quad u^2-v^2 = \frac{-1}{ab}$

$$\frac{1}{ab}R = -1+2vx+(u^2-v^2)x^2 = x^2(u^2-v^2+2vx^{-1}-x^{-2}) = x^2(u^2-(v-x^{-1})^2)$$

$$= u^2x^2(1-u^{-2}(v-x^{-1})^2) = u^2x^2(1-t^2) \quad\text{where } t = u^{-1}(v-x^{-1})$$

So $\sqrt{R} = \sqrt{ab}\,ux\sqrt{1-t^2}$, $x\sqrt{R} = \sqrt{ab}\,ux^2\sqrt{1-t^2}$; $\frac{dx}{x\sqrt{R}}$

$$= \frac{ux^2\,dt}{\sqrt{ab}\,ux^2\sqrt{1-t^2}} = \frac{1}{\sqrt{ab}}\,\frac{dt}{\sqrt{1-t^2}}.\quad \int_a^b\frac{dx}{x\sqrt{R}} = \int_{-1}^{1}\frac{1}{\sqrt{ab}}\,\frac{dt}{\sqrt{1-t^2}}$$

$dt = \frac{dx}{ux^2}$
$dx = ux^2\,dt$
@ $x=a$ $\quad t=-1$
@ $x=b$ $\quad t=1$

$= \frac{\pi}{\sqrt{ab}}$. Thus $\displaystyle\int_a^b -\frac{ab}{x\sqrt{R}}\,dx = \boxed{-\pi\sqrt{ab}}$

$$\int_a^b \frac{\sqrt{R}}{2\pi x}\,dx$$

$$= \frac{1}{2\pi}\left\{\boxed{1}+\boxed{2}+\boxed{3}\right\} = \frac{1}{2}\left(\frac{a+b}{2}-\sqrt{ab}\right) = \frac{1}{4}(b-2\sqrt{ab}+a) = \left(\frac{\sqrt{b}-\sqrt{a}}{2}\right)^2$$

$$\sqrt{a} = \begin{cases} 1-\sqrt{c} & c\le 1 \\ \sqrt{c}-1 & c>1 \end{cases},\quad \sqrt{b} = 1+\sqrt{c}.\quad \frac{\sqrt{b}-\sqrt{a}}{2} = \begin{cases} \sqrt{c} & c\le 1 \\ 1 & c>1 \end{cases}.$$

$$\boxed{\left(\frac{\sqrt{b}-\sqrt{a}}{2}\right)^2 = \begin{cases} c & c\le 1 \\ 1 & c>1 \end{cases}}$$