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Transcendental values of class group L-functions

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Abstract Let K be an algebraic number field and f a complex-valued function on the ideal class group of K. Then, f extends in a natural way to the set of all non-zero ideals of the ring of integers of K and we can consider the Dirichlet series $L(s, f) = \sum_{\mathfrak{a}} f(\mathfrak{a})\mathbf{N}(\mathfrak{a})^{-s}$ which converges for $\Re(s) > 1$. After extending this function to $\Re(s) = 1$, and in the case that f does not contain the trivial character, we study the special value $L(1, f)$ when f is algebraic valued and K is an imaginary quadratic field. Applying Kronecker's limit formula and Baker's theory of linear forms in logarithms, we derive a variety of results related to the transcendence of this special value.

1 Introduction

Let K be an algebraic number field, f a $\mathbb{C}$-valued function of the ideal class group $\mathcal{H}_K$ of K. We consider the Dirichlet series

$$L(s, f) := \sum_{\mathfrak{a}} \frac{f(\mathfrak{a})}{\mathbf{N}(\mathfrak{a})^s}, \tag{1}$$

where the summation is over all integral ideals $\mathfrak{a}$ of the ring of integers, $\mathcal{O}_K$, of K. If f is identically 1, then $L(s, f)$ is the Dedekind zeta function of K. If f is a character

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χ of the ideal class group $\mathcal{H}_K$ of K, then, $L(s, \chi)$ is a Hecke L-function. Our goal in this paper is to investigate special values of $L(s, f)$ at $s = 1$ in the case that K is an imaginary quadratic field and f is $\overline{\mathbb{Q}}$-valued.

This study will reveal new aspects of the transcendental nature of $L(1, \chi)$ when χ is an ideal class character. In particular, we will show that the values $L(1, \chi)$ are linearly independent over $\overline{\mathbb{Q}}$ as χ ranges over non-trivial characters of the ideal class group modulo complex conjugation.

We first prove:

Theorem 1 *$L(s, f)$ extends analytically for all $s \in \mathbb{C}$ except possibly at $s = 1$ where it has a simple pole with residue*

$$\rho_f := \sum_{\mathfrak{a} \in \mathcal{H}_K} f(\mathfrak{a}).$$

The series (1) converges at $s = 1$ if and only if $\rho_f = 0$.

Thus, in the case that the series converges at $s = 1$, it makes sense to consider $L(1, f)$. By a deeper analysis, we will show:

Theorem 2 *Let K be an imaginary quadratic field and $f : \mathcal{H}_K \to \overline{\mathbb{Q}}$ be not identically zero. Suppose that $\rho_f = 0$. Then, $L(1, f) \neq 0$ unless $f(\mathfrak{C}) + f(\mathfrak{C}^{-1}) = 0$ for every ideal class $\mathfrak{C} \in \mathcal{H}_K$. Moreover, $L(1, f)/\pi$ is a $\overline{\mathbb{Q}}$-linear combination of logarithms of algebraic numbers. In particular, $L(1, f)/\pi$ is transcendental whenever $L(1, f) \neq 0$.*

This result has several interesting corollaries.

Corollary 3 *Let K be an imaginary quadratic field and χ a non-trivial character of the ideal class group of K. Then, $L(1, \chi)/\pi$ is a non-zero $\overline{\mathbb{Q}}$-linear combination of logarithms of algebraic numbers and hence transcendental.*

Since complex conjugation acts on the group of ideal class characters we see by a simple calculation that $L(1, \chi) = L(1, \overline{\chi})$ for any ideal class character χ. We denote by $\mathcal{H}_K^*$ a set of orbit representatives under this action.

Corollary 4 *Let K be an imaginary quadratic field and $\mathcal{H}_K$ its ideal class group. The values $L(1, \chi)$ (as χ ranges over the non-trivial characters of $\mathcal{H}_K^*$) and π are linearly independent over $\overline{\mathbb{Q}}$.*

Thus, in the special case that the ideal class group $\mathcal{H}_K$ is an elementary abelian 2-group, the corollary implies that the $L(1, \chi)$ as χ ranges over the non-trivial characters of $\mathcal{H}_K$, are linearly independent over $\overline{\mathbb{Q}}$.

In the general case, writing

$$f = \sum_{\chi \neq 1} c_\chi \chi,$$

we easily see that the condition $f(\mathfrak{C})+f(\mathfrak{C}^{-1})$ for every class $\mathfrak{C}$ implies that $c_\chi + c_{\overline{\chi}} = 0$ for every $\chi \neq 1$. The linear independence of the $L(1,\chi)$ over $\overline{\mathbb{Q}}$ as χ ranges over the non-trivial characters of $\mathcal{H}_K^*$ is then clear from Theorem 2. To deduce linear independence along with π, we proceed as follows. Suppose that the $L(1,\chi)$ and π are linearly dependent over $\overline{\mathbb{Q}}$:

$$\sum_{\chi\in\mathcal{H}_K^*} \alpha_\chi L(1,\chi) + \beta\pi = 0,$$

with $\alpha_\chi, \beta \in \overline{\mathbb{Q}}$, $\beta \neq 0$. Then, setting

$$f = -\frac{1}{\beta}\sum_{\chi\in\mathcal{H}_K^*} \alpha_\chi \chi,$$

we have $L(1, f) = \pi$. By Theorem 2, we derive a contradiction.

Theorem 2 implies that at most one such $L(1,\chi)$ is algebraic:

Corollary 5 *All of the values $L(1,\chi)$ as χ ranges over the non-trivial characters of $\mathcal{H}_K^*$, with at most one exception, are transcendental.*

The elimination of this singular possibility, in other words, the proof of transcendence of $L(1,\chi)$ for all non-trivial χ seems difficult and is related to Schanuel's conjecture. Indeed, a "weaker" version of Schanuel suffices for our purposes. This is the conjecture that logarithms of algebraic numbers which are linearly independent over $\mathbb{Q}$ are algebraically independent. Assuming the "weaker" Schanuel's conjecture, one can show the transcendence of $L(1,\chi)$ for all non-trivial χ (see remarks at end as well as [8]).

When χ is a genus character, one can relate $L(1,\chi)$ to classical Dirichlet L-functions attached to quadratic characters [18]. Utilising this connection, we will prove the following.

Theorem 6 *Let K be an imaginary quadratic field with character χ_D. Then,*

$$\exp\left(\frac{L'(1,\chi_D)}{L(1,\chi_D)} - \gamma\right)$$

and π are algebraically independent. Here γ is Euler's constant.

Thus, we have from the above theorem that

$$\exp\left(\frac{L'(1,\chi_D)}{L(1,\chi_D)} - \gamma\right)$$

is transcendental. In particular,

$$\frac{L'(1,\chi_D)}{L(1,\chi_D)} \neq \gamma,$$

for any D. More generally, we have:

Corollary 7

$$\frac{L'(1,\chi_D)}{L(1,\chi_D)} - \gamma$$

is not a $\mathbb{Q}$-linear combination of logarithms of algebraic numbers.

From the theorem, we can also deduce the following curious corollary.

Corollary 8 *If for some D, we have $L'(1,\chi_D) = 0$, then e^{γ} is transcendental.*

It is unlikely that such a D exists for a variety of reasons. But this seems difficult to prove. We consider a related question, namely, the vanishing of $L'(1,\chi)$ for any Dirichlet character χ (mod q). In this direction, we use results of Ihara et al. [10] to prove:

Theorem 9 *For q prime, the number of $\chi \neq \chi_0$ (mod q) for which $L'(1,\chi) = 0$ is $O(q^{\epsilon})$ for any $\epsilon > 0$.*

This last result is very analytic in flavour and it is unlikely that one can show the non-vanishing of $L'(1,\chi)$ in general using analytic methods. Theorem 6 allows us to connect this question to special values of the Γ-function via the Chowla–Selberg formula. Indeed, our proof of Theorem 6 leads to a simple proof of the Chowla–Selberg formula which we give in Sect. 7. Naturally, this leads one to enquire about the transcendence of special values of the Γ-function. Not much is known in this context. It is well-known that $\Gamma(1/2) = \sqrt{\pi}$ is transcendental by the celebrated theorem of Lindemann. The transcendence of $\Gamma(1/3)$ and $\Gamma(1/4)$ was established by Chudnovsky [3] in 1976. Recently, Grinspan [6] and Vasilev [19] independently showed that at least two of the three numbers π, $\Gamma(1/5)$, $\Gamma(2/5)$ are algebraically independent. Very likely, all of the three numbers are algebraically independent. Apart from these results, no further results are known regarding the transcendence of the Γ-function at rational, non-integral arguments. Thus, in this context, the following theorem is of interest.

Theorem 10 *Let $q > 1$ and $q|24$. Let a be coprime to q. There exists a finite set S and a collection of pair-wise non-isogenous CM elliptic curves E_j, $j \in S$ defined over $\overline{\mathbb{Q}}$ with fundamental real periods ω_j such that $\Gamma(a/q)$ lies in the field generated over $\mathbb{Q}$ by π and the ω_j. In particular, if π and the ω_j's are algebraically independent, then $\Gamma(a/q)$ is transcendental.*

The key point here is that the non-trivial Dirichlet characters (mod 24) are all quadratic. Consequently, one can use the Chowla–Selberg formula (as we have stated it below) to express $\Gamma(a/q)$ as a product of π and periods of various non-isogenous elliptic curves.

Before moving on in our discussion, we observe an amusing corollary of the above theorem:

Corollary 11 *All of the numbers*

$$\Gamma(1/8), \Gamma(3/8), \Gamma(5/8), \Gamma(7/8)$$

are transcendental with at most one exception.

Schanuel's conjecture predicts that if $x_1, \ldots, x_n$ are linearly independent over $\mathbb{Q}$, then the transcendence degree of the field

$$\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$$

is at least n. The previous theorem motivates the following extension of Schanuel's conjecture. Suppose that $x_1, \ldots, x_n$ are linearly independent over $\overline{\mathbb{Q}}$. Let $\wp_2, \ldots, \wp_n$ be the Weierstrass $\wp$-functions attached to non-isogenous CM elliptic curves $E_2, \ldots, E_n$ defined over $\overline{\mathbb{Q}}$. If $x_2, \ldots, x_n$ are not contained in the poles of the $\wp_i$, $2 \leq i \leq n$, then, the transcendence degree of the field

$$\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \wp_2(x_2), \ldots, \wp_n(x_n))$$

is at least n. Thus, choosing $x_1 = \pi i$ and $x_j = \omega_j/2$ with the ω_j as in Theorem 10, the conjecture allows us to deduce that π and the ω_j's are algebraically independent.

Our conjecture is a special case of a more general conjecture of Grothendieck (see [5]). This conjecture asserts that the transcendence degree of the field generated by the periods of an algebraic variety is equal to d where d is the dimension of the Hodge group of the variety. In our case, we consider the variety

$$X = \mathbb{P}^1 \times E_2 \times \cdots \times E_n$$

where E_i are pairwise non-isogenous elliptic curves with complex multiplication. The Hodge group of $H^2(\mathbb{P}^1) \otimes \cdots \otimes H^1(E_n)$ is isomorphic to

$$\mathbb{G}_m^1 \times \prod_{i=2}^{n} (R_{K_i/\mathbb{Q}}\mathbb{G}_m)^1,$$

where K_i is the imaginary quadratic field corresponding to E_i and the superscript denotes elements of norm 1. The dimension of this group is n.

The study of $L(s, f)$ allows us to give a simple proof of the Chowla–Selberg formula [2]. This then facilitates an application of Nesterenko's theorem to deduce the transcendence result indicated in Theorem 6.

One could discuss a more general situation where we consider functions f defined on ray class groups and similar formulas and results can be derived. However, these derivations are a bit more complicated and their treatment would alter the elegance and simplicity of this paper. Therefore, we have decided to treat the general case in a future paper.

2 Preliminaries

In the discussion below, a pivotal role will be played by the fundamental theorem of Baker concerning linear forms in logarithms. Let us note that here and later, we interpret log as the principal value of the logarithm with the argument lying in the interval $(-\pi, \pi]$. We record Baker's theorem below.

Lemma 12 *If* $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}} \backslash \{0\}$ *and* $\beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}}$, *then*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental. The latter case arises if $\log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over* $\mathbb{Q}$ *and* $\beta_1, \ldots, \beta_n$ *are not all zero.*

Proof This is the content of Theorems 2.1 and 2.2 of [1]. □

We will also use Kronecker's limit formula as discussed in the works of Siegel [18], Ramachandra [16] and Lang [11]. We begin by reviewing this.

Let $\Delta(z)$ be Ramanujan's cusp form:

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i z}.$$

Now let K be an imaginary quadratic field and let $\mathfrak{b}$ be an ideal of $\mathcal{O}_K$. If $[\beta_1, \beta_2]$ is an integral basis of $\mathfrak{b}$ with $\Im(\beta_2/\beta_1) > 0$, we define

$$g(\mathfrak{b}) = (2\pi)^{-12} (\mathbf{N}(\mathfrak{b}))^6 \Delta(\beta_2/\beta_1).$$

One can verify (as on page 109 of [16]) that $g(\mathfrak{b})$ is well-defined and does not depend on the choice of integral basis of $\mathfrak{b}$. In fact, by Lemma 2 of [16] (or page 280 of [11]), $g(\mathfrak{b})$ depends only on the ideal class $\mathfrak{b}$ belongs to in the ideal class group. Thus, if $\mathfrak{C}$ is an ideal class, we write $g(\mathfrak{C})$ for the common value $g(\mathfrak{b})$ as $\mathfrak{b}$ ranges over the elements of the class $\mathfrak{C}$

Let d_K be the discriminant of K and w denote the number of roots of unity in $\mathcal{O}_K$. Writing

$$\zeta(s, \mathfrak{C}) = \sum_{\mathfrak{a} \in \mathfrak{C}} \frac{1}{\mathbf{N}(\mathfrak{a})^s},$$

for the ideal class zeta function, we have by Kronecker's limit formula

$$\zeta(s, \mathfrak{C}) = \frac{2\pi}{w\sqrt{|d_K|}} \left(\frac{1}{s-1} + 2\gamma - \log |d_K| - \frac{1}{6} \log |g(\mathfrak{C}^{-1})| \right) + O(s-1), \quad (2)$$

as $s \to 1^+$. (Note that there is a sign error in formula (2) on page 280 of [11].)

Proposition 13 *If $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are ideal classes, then $g(\mathfrak{C}_1)/g(\mathfrak{C}_2)$ is an algebraic number lying in the Hilbert class field of K.*

Proof This follows immediately from Lemma 3 of [2] and is a classical result from the theory of complex multiplication. □

Proposition 14 *For any ideal $\mathfrak{b}$, $g(\mathfrak{b})/g(\mathcal{O}_K)$ is an element lying in the Hilbert class field K_H of K. If $\mathfrak{p}$ is a prime ideal of K and $\sigma_{\mathfrak{p}}$ is the Frobenius automorphism in* $\mathrm{Gal}(K_H/K)$*, then*

$$\sigma_{\mathfrak{p}}\left(g(\mathfrak{b})/g(\mathcal{O}_K)\right) = g(\mathfrak{p}^{-1}\mathfrak{b})/g(\mathfrak{p}^{-1}\mathcal{O}_K), \quad \overline{g(\mathfrak{b})/g(\mathcal{O}_K)} = g(\mathfrak{b}^{-1})/g(\mathcal{O}_K).$$

Proof The first part is the content of Theorem 1 on page 161 of [11]. The action of complex conjugation is easily deduced from the equation $\overline{j(\mathfrak{b})} = j(\bar{\mathfrak{b}})$ for the j-function. □

In the later sections of the paper, we will make fundamental use of a result of Nesterenko [15].

Proposition 15 *For any imaginary quadratic field with discriminant D and character χ_D, the numbers π, $e^{\pi\sqrt{D}}$ and*

$$\prod_{a=1}^{D} \Gamma(a/D)^{\chi_D(a)},$$

are algebraically independent over $\overline{\mathbb{Q}}$.

Proof See Corollary 3.2, page 6 of [15]. □

3 Proofs of Theorem 1 and Corollary 3

We first write

$$L(s, f) = \sum_{\mathfrak{C}\in\mathcal{H}_K} f(\mathfrak{C})\zeta(s, \mathfrak{C}).$$

Each $\zeta(s, \mathfrak{C})$ extends to all $s \in \mathbb{C}$ with the exception of $s = 1$, where it has a simple pole with residue

$$\frac{2^{r_1}(2\pi)^{r_2} R_K}{w\sqrt{|d_K|}},$$

where r_1 is the number of real embeddings, $2r_2$ is the number of complex embeddings and R_K is the regulator of K. We conclude that $L(s, f)$ extends analytically to all $s \in \mathbb{C}$ apart from a simple pole at $s = 1$ with residue

$$\frac{2^{r_1}(2\pi)^{r_2} R_K}{w\sqrt{|d_K|}} \sum_{\mathfrak{C}} f(\mathfrak{C}).$$

Thus, $L(s, f)$ is analytic at $s = 1$ if and only if $\rho_f = 0$. To study the convergence of the Dirichlet series $L(s, f)$ at $s = 1$, we proceed as follows. The number of ideals with norm $\leq x$ and lying in a fixed class $\mathfrak{C}$ is well-known to be (see [12]),

$$\frac{2^{r_1}(2\pi)^{r_2}R_K}{w\sqrt{|d_K|}}x + O\left(x^{\frac{d}{d+1}}\right),$$

where d is the degree of K over $\mathbb{Q}$. Letting

$$S(x) = \sum_{\mathbf{N}(\mathfrak{a})\leq x} f(\mathfrak{a}),$$

we have by the general technique of partial summation (see p. 17 of [13]) that

$$L(s, f) = s\int_1^\infty \frac{S(x)}{x^{s+1}}dx,$$

for $\Re(s) > 1$. Now,

$$S(x) = \sum_{\mathfrak{C}\in\mathcal{H}_K}\sum_{\mathfrak{a}\in\mathfrak{C},\mathbf{N}(\mathfrak{a})\leq x} f(\mathfrak{a}) = \sum_{\mathfrak{C}\in\mathcal{H}_K} f(\mathfrak{C})\left(\frac{2^{r_1}(2\pi)^{r_2}R_K}{w\sqrt{|d_K|}}x + O\left(x^{\frac{d}{d+1}}\right)\right)$$

which is easily seen to be

$$\frac{2^{r_1}(2\pi)^{r_2}R_K\rho_f}{w\sqrt{|d_K|}}x + O\left(x^{\frac{d}{d+1}}\right).$$

Hence, by partial summation, it follows immediately that $L(s, f)$ converges at $s = 1$ if and only if $\rho_f = 0$. This completes the proof of Theorem 1.

To prove Corollary 3, we begin by noting that in the case K is an imaginary quadratic field, the formulas become simple and we can apply Kronecker's limit formula. In this situation, when the series converges, we have by (2),

$$\frac{L(1, f)}{\pi} = \frac{-1}{3w\sqrt{|d_K|}}\sum_{\mathfrak{C}\in\mathcal{H}_K} f(\mathfrak{C})\log|g(\mathfrak{C}^{-1})|. \tag{3}$$

Now we invoke Proposition 13. Indeed, by this proposition, we have for the identity class $\mathfrak{C}_0$, that $g(\mathfrak{C}^{-1})/g(\mathfrak{C}_0)$ is algebraic. Thus, as $\rho_f = 0$, we have

$$\frac{L(1, f)}{\pi} = \frac{-1}{3w\sqrt{|d_K|}}\sum_{\mathfrak{C}\in\mathcal{H}_K} f(\mathfrak{C})\log|g(\mathfrak{C}^{-1})/g(\mathfrak{C}_0)|, \tag{4}$$

for any fixed class $\mathfrak{C}_0$ of $\mathcal{H}_K$.

Specializing to the case $f = \chi$, where χ is a non-trivial character of the ideal class group $\mathcal{H}_K$, and using the celebrated theorem that $L(1, \chi) \neq 0$, we deduce Corollary 3 by virtue of Lemma 12. This completes the proofs.

4 Proofs of Theorem 2 and Corollaries 4 and 5

In view of (4), and Lemma 12 (Baker's theorem), the only part of Theorem 2 that remains to be proved is the non-vanishing of $L(1, f)$. To this end, we will require three lemmas.

Lemma 16 *Let K be an imaginary quadratic field and $f : \mathcal{H}_K \to \overline{\mathbb{Q}}$. Then, $L(1, f) = 0$ implies that $L(1, f^\sigma) = 0$ for any Galois automorphism σ of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Proof Equation (4) expresses $L(1, f)/\pi$ as a linear form of logarithms of algebraic numbers. Now choose a maximal set of $\mathbb{Q}$ linearly independent numbers from $\{\log |g(\mathfrak{C}^{-1})/g(\mathfrak{C}_0)| : \mathfrak{C} \in \mathcal{H}_K\}$. Denote this set by $\log \alpha_1, \ldots, \log \alpha_t$. Thus,

$$\log |g(\mathfrak{C}^{-1})/g(\mathfrak{C}_0)| = \sum_{j=1}^{t} x(\mathfrak{C}, j) \log \alpha_j,$$

where the $x(\mathfrak{C}, j)$'s are rational numbers. Hence

$$L(1, f) = -\frac{\pi}{3w\sqrt{|d_K|}} \sum_{j=1}^{t} \sum_{\mathfrak{C} \in \mathcal{H}_K} f(\mathfrak{C}) x(\mathfrak{C}, j) \log \alpha_j.$$

If $L(1, f) = 0$, then an application of Baker's theorem (Lemma 12) gives that

$$\sum_{\mathfrak{C} \in \mathcal{H}_K} f(\mathfrak{C}) x(\mathfrak{C}, j) = 0, \quad j = 1, 2, \ldots, t.$$

Since the $x(\mathfrak{C}, j)$'s are rational numbers, we deduce that for every Galois automorphism σ,

$$\sum_{\mathfrak{C} \in \mathcal{H}_K} f^\sigma(\mathfrak{C}) x(\mathfrak{C}, j) = 0, \quad j = 1, 2, \ldots, t.$$

Consequently, $L(1, f^\sigma) = 0$. □

The next lemma allows us to reduce the proof of Theorem 2 to the case when f is rational valued.

Lemma 17 *Let M be the algebraic number field generated by the values of f. Let $r = [M : \mathbb{Q}]$ and choose a basis $\beta_1, \ldots, \beta_r$ of M over $\mathbb{Q}$ and write*

$$f(\mathfrak{C}) = \sum_{i=1}^{r} \beta_i f_i(\mathfrak{C}),$$

with $f_i(\mathfrak{C})$ rational. Then, $L(1, f) = 0$ implies $L(1, f_i) = 0$ for $i = 1, 2, \ldots, r$.

Proof Let $M = M^{(1)}, \ldots, M^{(r)}$ be the conjugate fields of M. The map $x \to x^{(j)}$ from M to $M^{(j)}$ extends to a Galois automorphism σ_j of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus,

$$f^{\sigma_j}(\mathfrak{C}) = \sum_{i=1}^{r} \beta_i^{(j)} f_i(\mathfrak{C}).$$

Clearly, the matrix $(\beta_i^{(j)})$ is invertible since $\beta_1, \ldots, \beta_r$ is a basis, and we can express $f_i(\mathfrak{C})$ as a $\overline{\mathbb{Q}}$-linear combination of the $f^{\sigma_j}(\mathfrak{C})$. By Lemma 16, we have that $L(1, f) = 0$ implies $L(1, f^{\sigma_j}) = 0$ for every j. Thus, $L(1, f_i) = 0$ for $1 \le i \le r$, as desired. □

Lemma 18 *If f is rational-valued and $L(1, f) = 0$, then $f(\mathfrak{C}) + f(\mathfrak{C}^{-1}) = 0$ for every ideal class $\mathfrak{C}$.*

Proof If $L(1, f) = 0$, then

$$\sum_{\mathfrak{C} \in \mathcal{H}_K} f(\mathfrak{C}) \log |g(\mathfrak{C}^{-1})/g(\mathfrak{C}_0)| = 0.$$

Clearing denominators, we may suppose that f is integer-valued. Exponentiating the above expression gives

$$\prod_{\mathfrak{C} \in \mathcal{H}_K} \left| \frac{g(\mathfrak{C}^{-1})}{g(\mathfrak{C}_0)} \right|^{f(\mathfrak{C})} = 1.$$

To remove the absolute values, we square the expression and pair up $\mathfrak{C}$ with $\mathfrak{C}^{-1}$ and re-arrange it to deduce that

$$\prod_{\mathfrak{C}} \left[\frac{g(\mathfrak{C})}{g(\mathfrak{C}_0)} \right]^{f(\mathfrak{C}) + f(\mathfrak{C}^{-1})} = 1.$$

Each of the factors in the product is an algebraic number and applying Proposition 14, we see that

$$\prod_{\mathfrak{C}} \left[\frac{g(\mathfrak{p}^{-1}\mathfrak{C}^{-1})}{g(\mathfrak{p}^{-1}\mathfrak{C}_0)} \right]^{f(\mathfrak{C}) + f(\mathfrak{C}^{-1})} = 1,$$

for any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$. Taking absolute values and then logarithms, we conclude that

$$\sum_{\mathfrak{C}} (f(\mathfrak{C}) + f(\mathfrak{C}^{-1})) \log |g(\mathfrak{p}^{-1}\mathfrak{C}^{-1})/g(\mathfrak{p}^{-1}\mathfrak{C}_0)| = 0,$$

for every prime ideal $\mathfrak{p}$. By the Chebotarev density theorem, the $\mathfrak{p}^{-1}$'s range over all elements of $\mathcal{H}_K$ as $\mathfrak{p}$ ranges over all prime ideals of $\mathcal{O}_K$.

We view these equations as a matrix equation

$$DF = 0$$

where F is the transpose of the row vector $(f(\mathfrak{C}^{-1}))_{\mathfrak{C}\in H_K}$ and D is the "Dedekind–Frobenius" matrix whose (i, j)th entry is given by $\log g(\mathfrak{C}_i^{-1}\mathfrak{C}_j)/g(\mathfrak{C}_i^{-1})$ with $\mathfrak{C}_i, \mathfrak{C}_j$ running over the elements of the ideal class group. The first column of D is the zero vector and we can re-write our matrix equation as

$$D_0 F_0 = 0$$

where F_0 is the transpose of the row vector $(f(\mathfrak{C}^{-1}))_{\mathfrak{C}\neq 1}$ and D_0 is the matrix obtained from D by deleting the row and column corresponding to the identity element. By the theory of the Dedekind–Frobenius determinant (see for example, p. 71 of [20]), the determinant of D_0 is

$$\prod_{\chi\neq 1}\left(\sum_{\mathfrak{a}} \chi(\mathfrak{a}) \log g(\mathfrak{a}^{-1})\right) \neq 0,$$

since by formula (3), each factor is up to a non-zero scalar, $L(1, \chi)$, which is non-zero. Thus, $f(\mathfrak{C}) + f(\mathfrak{C}^{-1}) = 0$ for all $\mathfrak{C} \neq \mathfrak{C}_0$. Since

$$\sum_{\mathfrak{C}} f(\mathfrak{C}) + f(\mathfrak{C}^{-1}) = 0,$$

we deduce that $f(\mathfrak{C}_0) + f(\mathfrak{C}_0^{-1}) = 2f(\mathfrak{C}_0) = 0$ as well. This completes the proof. □

The proof of Theorem 2 can now be given as follows. First, if f is rational-valued, we are done by the previous lemma. Lemma 17 allows us to reduce to the rational-valued case. This completes the proof.

Corollary 4 is also now immediate. Indeed, suppose that

$$\sum_{\chi\neq 1, \chi\in\mathcal{H}_K^*} c_\chi L(1, \chi) \in \overline{\mathbb{Q}}\pi,$$

for some $c_\chi \in \overline{\mathbb{Q}}$. Then, setting $f = \sum_{\chi\neq 1, \chi\in\mathcal{H}_K^*} c_\chi \chi$ we have $L(1, f)/\pi$ is algebraic. Since $\rho_f = 0$, we can apply Theorem 2 and deduce that f is identically zero. By the independence of characters, this means that each c_χ is zero.

Corollary 5 follows directly from Corollary 4 since two algebraic numbers are linearly dependent over $\overline{\mathbb{Q}}$.

5 Genus characters and $L'(1, \chi_D)$

As before, let K be an imaginary quadratic field with discriminant $D < 0$. Real-valued characters of the ideal class group of K are called genus characters. These characters can be extended to functions on the ideal classes of $\mathcal{O}_K$ in the obvious way. Such extended characters take on only the values $0, \pm 1$. By a classical theorem of Kronecker, they have a simple description. For each factorization $D = D_1 D_2$ with D_1, D_2 being fundamental discriminants, we define a character χ_{D_1,D_2} by setting it to be

$$\chi_{D_1,D_2}(\mathfrak{p}) = \begin{cases} \chi_{D_1}(\mathbf{N}(\mathfrak{p})) & \text{if } (\mathfrak{p}, D_1) = 1 \\ \chi_{D_2}(\mathbf{N}(\mathfrak{p})) & \text{if } (\mathfrak{p}, D_2) = 1. \end{cases}$$

One can show that this is well-defined and that it defines a character on the ideal class group. We refer the reader to page 60 of [18] for the background on genus characters. We have the Kronecker factorization formula:

$$L(s, \chi_{D_1,D_2}) = L(s, \chi_{D_1}) L(s, \chi_{D_2}).$$

Corresponding to the factorization $D = 1 \cdot D$, we get

$$L(s, \chi_{1,D}) = \zeta(s) L(s, \chi_D).$$

The left hand side is $\zeta_K(s)$ and so we can write

$$\sum_{\mathfrak{C} \in \mathcal{H}_K} \zeta(s, \mathfrak{C}) = \zeta(s) L(s, \chi_D).$$

This identity could have been derived in other ways. Applying the Kronecker limit formula, and comparing the constant term in the Laurent expansion of both sides, we obtain as in [18]:

Proposition 19

$$\gamma L(1, \chi_D) + L'(1, \chi_D) = \frac{2\pi}{w\sqrt{|d_K|}} \sum_{\mathfrak{C} \in \mathcal{H}_K} \left(2\gamma - \log |d_K| - \frac{1}{6} \log |g(\mathfrak{C}^{-1})| \right).$$

Using Dirichlet's class number formula, we deduce:

Corollary 20

$$\frac{L'(1, \chi_D)}{L(1, \chi_D)} = \gamma - \log |d_K| - \frac{1}{6h} \sum_{\mathfrak{C} \in \mathcal{H}_K} \log |g(\mathfrak{C})|,$$

where h denotes the order of $\mathcal{H}_K$.

In particular, we deduce the following interesting result.

Theorem 21 *For any ideal class* $\mathfrak{C}$*,*

$$\frac{L'(1,\chi_D)}{L(1,\chi_D)} - \gamma + \frac{1}{6}\log|g(\mathfrak{C})|$$

is a $\overline{\mathbb{Q}}$*-linear combination of logarithms of algebraic numbers.*

6 Proof of Theorem 6

We now analyze the asymptotic behaviour of the formula in Corollary 20 using the theory of Hurwitz zeta functions. As a result, we derive a simple proof of the Chowla–Selberg formula. Recall that the Hurwitz zeta function $\zeta(s,x)$ is defined by the series

$$\zeta(s,x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.$$

This series converges for $\Re(s) > 1$ and Hurwitz showed how one can extend it to the entire complex plane apart from $s = 1$ where it has a simple pole with residue 1. Given a Dirichlet character χ mod q, we can write

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = q^{-s} \sum_{a=1}^{q} \chi(a)\zeta(s,a/q).$$

Thus,

$$L'(s,\chi) = -(\log q)q^{-s} \sum_{a=1}^{q} \chi(a)\zeta(s,a/q) + q^{-s} \sum_{a=1}^{q} \chi(a)\zeta'(s,a/q).$$

Using the well-known formulas

$$\zeta(0,x) = \frac{1}{2} - x, \qquad \zeta'(0,x) = \log(\Gamma(x)/2\pi),$$

where the differentiation is with respect to the s-variable, we deduce that

$$L(0,\chi) = \sum_{a=1}^{q} \chi(a)(1/2 - a/q),$$

and

$$L'(0,\chi) = -(\log q)L(0,\chi) + \sum_{a=1}^{q} \chi(a)\log\Gamma(a/q), \tag{5}$$

since $\sum_{a=1}^{q} \chi(a) = 0$. If χ is odd and primitive, $L(s, \chi)$ satisfies a functional equation of the form

$$A^s \Gamma((s+1)/2) L(s, \chi) = \varpi A^{1-s} \Gamma((2-s)/2) L(1-s, \overline{\chi}),$$

where ϖ (called the root number) is a complex number (see page 71 of [4]) and $A = \sqrt{q/\pi}$. We also recall that for quadratic characters χ, the root number ϖ is 1. We logarithmically differentiate this expression to obtain:

$$\log A + \frac{1}{2}\psi((s+1)/2) + \frac{L'}{L}(s, \chi) = -\log A - \frac{1}{2}\psi((2-s)/2) - \frac{L'}{L}(1-s, \overline{\chi}),$$

where $\psi(s)$ denotes the digamma function, which is the logarithmic derivative of the gamma function. Putting $s = 1$ into the formula, and using (see, for example, p. 301 of [14])

$$\psi(1) = -\gamma, \qquad \psi(1/2) = -\gamma - 2\log 2$$

we deduce

$$\frac{L'}{L}(1, \chi) = -2\log A + \gamma + \log 2 - \frac{L'}{L}(0, \overline{\chi}). \tag{6}$$

Now we specialize our discussion to quadratic characters associated to an imaginary quadratic field K. Such a character is necessarily odd and if K has discriminant D, then this character, which we denote by χ_D is a primitive character modulo D. In this situation, we have from the functional equation

$$L(0, \chi_D) = 2h_D/w_D,$$

where h_D and w_D denote the class number and number of roots of unity of K. Thus, injecting formula (5) into equation (6), we get on exponentiating,

$$\exp\left(\frac{L'(1, \chi_D)}{L(1, \chi_D)} - \gamma\right) = (2D/A^2) \prod_{a=1}^{D} \Gamma(a/D)^{-\chi_D(a) w_D/2h_D}.$$

By Proposition 15, and the fact that $A = \sqrt{D/\pi}$, Theorem 6 is now immediate.

7 The Chowla–Selberg formula revisited

We can combine the calculation of the previous section with Corollary 20 to deduce the Chowla–Selberg formula:

$$\prod_{\mathfrak{C} \in \mathcal{H}_K} g(\mathfrak{C})^{1/3} = \left(\frac{1}{2\pi |d_K|}\right)^{2h_D} \prod_{a=1}^{D} \Gamma(a/D)^{w_D \chi_D(a)}.$$

We analyze the left hand side of this equation following [7]. Let E be an elliptic curve with complex multiplication by an order in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$. Formula (3) of [7] states that any period of E is up to an algebraic factor, given by the right hand side of the above equation. In other words,

$$f(\chi_D) := \prod_{a=1}^{D} \Gamma(a/D)^{\chi_D(a)}$$

is equal to a product of a power of π and a power of the period of the CM elliptic curve attached to the full ring of integers of $\mathbb{Q}(\sqrt{-D})$ (up to an algebraic factor).

More generally, we can define for any character χ (mod D),

$$f(\chi) = \prod_{a=1}^{D} \Gamma(a/D)^{\chi(a)}.$$

Let $q|D$ and χ be a real primitive character (mod q). Let χ^* denote the character obtained by extending χ to residue classes (mod D). Then, it is not hard to see that $f(\chi^*)$ is (up to a non-zero algebraic factor) equal to $f(\chi)$. Indeed, recall that

$$\Gamma(z)\Gamma(z+1/q)\cdots\Gamma(z+(q-1)/q) = q^{1/2-qz}(2\pi)^{(q-1)/2}\Gamma(qz).$$

Thus,

$$\begin{aligned} f(\chi^*) &= \prod_{a=1}^{D} \Gamma(a/D)^{\chi^*(a)} \\ &= \prod_{a=1}^{q} \Big[\Gamma(a/D)\Gamma((a+q)/D)\cdots\Gamma((a+(D/q-1)q)/D)\Big]^{\chi(a)} \\ &= \prod_{a=1}^{q} \Big[\Gamma(a/q)(2\pi)^{(D/q-1)/2}(D/q)^{1/2-a/q}\Big]^{\chi(a)} \\ &= f(\chi)(D/q)^{\sum_{a=1}^{q}(1/2-a/q)\chi(a)}. \end{aligned}$$

Since χ is a real character, the exponent of D/q is rational and so the second factor is algebraic and non-zero. Consequently, $f(\chi)$ and $f(\chi^*)$ are equal apart from a non-zero algebraic factor. We record this remark here since it will be used in the next section.

8 Proof of Theorem 10

We are now ready to prove Theorem 10. For each odd quadratic character χ_D, we have associated an imaginary quadratic extension k_D. Thus, $f(\chi)$ is defined for any odd quadratic character. We can associate a CM elliptic curve E_D, with ring of endomorphisms isomorphic to the ring of integers of k_D. Let ω_D be the real period of E_D. The Chowla–Selberg formula expresses

$$\sum_{a=1}^{D} \chi_D(a) \log \Gamma(a/D)$$

as a $\mathbb{Q}$-linear form in $\log \pi$, $\log \omega_D$ and the logarithm of a non-zero algebraic number. For any divisor q of 24, every non-trivial Dirichlet character mod q is quadratic. Noting that

$$\sum_{\chi \text{ even}} \chi(a) = \varphi(q)/2,$$

if $a \equiv \pm 1 (\bmod q)$ and zero otherwise, we deduce that

$$\sum_{\chi \text{ odd}} \chi(a) = \begin{cases} \varphi(q)/2 & \text{if } a \equiv 1 (\bmod q) \\ -\varphi(q)/2 & \text{if } a \equiv -1 (\bmod q) \\ 0 & \text{otherwise.} \end{cases}$$

By considering the combination

$$\prod_{\chi \text{ odd}} f(\chi)^{\chi(b)}$$

where the product is over odd characters (mod q), we find

$$\prod_{\chi \text{ odd}} f(\chi)^{\chi(b)} = \prod_{a=1}^{q} \Gamma(a/q)^{\sum_{\chi \text{ odd}} \chi(ab)}.$$

Since for any divisor q of 24, $b^2 \equiv 1 (\bmod q)$ for any b coprime to q and so we have $ab \equiv 1 (\bmod q)$ implies $a \equiv b (\bmod q)$. Thus,

$$\sum_{\chi \text{ odd}} \chi(ab) = \begin{cases} \varphi(q)/2 & \text{if } a \equiv b (\bmod q) \\ -\varphi(q)/2 & \text{if } a \equiv -b (\bmod q) \\ 0 & \text{otherwise.} \end{cases}$$

and we deduce that

$$\Gamma(a/q)\Gamma(1 - a/q)^{-1}$$

is the product of an algebraic number, a power of π and a product of powers of periods of non-isogenous elliptic curves. On the other hand,

$$\Gamma(a/q)\Gamma(1-a/q)$$

is a product of π and an algebraic number. Thus, we deduce that $\Gamma(a/q)$ is (up to an algebraic factor) a product of a power of π and periods of non-isogenous elliptic curves. This completes the proof.

To prove Corollary 11, we suppose that at least two of the numbers, $\Gamma(a/8)$, $\Gamma(b/8)$ (say), among

$$\Gamma(1/8), \Gamma(3/8), \Gamma(5/8), \Gamma(7/8)$$

are algebraic. By the proof of the previous theorem, we can write each term as a product of powers of π and periods ω_1 and ω_2 of two non-isogenous CM elliptic curves. By taking appropriate powers of $\Gamma(a/8)$, $\Gamma(b/8)$, we deduce that their quotient, which is algebraic, is a product of powers of π and ω_1. By a result of Chudnovsky [3], we know that π and ω_1 are algebraically independent. This completes the proof.

9 Proof of Theorem 9

As noted in Sect. 1, Ihara et al. proved the following theorem in [10].

Proposition 22 *Let* $\Lambda_0(1) = 1$ *and* $\Lambda_0(n) = 0$ *for* $n > 1$. *Define for* $k \geq 1$,

$$\Lambda_k(n) = \sum_{n_1 \cdots n_k = n} \Lambda(n_1) \cdots \Lambda(n_k),$$

where Λ *denotes the von Mangoldt function. Set*

$$\mu^{(a,b)} := \sum_{n=1}^{\infty} \frac{\Lambda_a(n)\Lambda_b(n)}{n^2}.$$

Then, for q prime and any $\epsilon > 0$

$$T_{a,b} := \sum_{\chi \neq \chi_0} P^{(a,b)}\left(\frac{L'}{L}(1,\chi)\right) = (-1)^{a+b}\mu^{(a,b)}\varphi(q) + O(q^{\epsilon}),$$

where $P^{(a,b)}(z) = z^a \bar{z}^b$.

Proof This is Theorem 5 of [10]. □

Remark It is easy to see that the series for $\mu^{(a,b)}$ converges. Indeed, $\Lambda(n) \leq \log n$ so that $\Lambda_k(n) \leq d_k(n)(\log n)^k$, where $d_k(n)$ denotes the number of factorizations of n as a product of k natural numbers. Consequently, $\Lambda_k(n) = O(n^{\epsilon})$ for any $\epsilon > 0$.

We apply the previous proposition with $a = b = k$ and $a = b = 2k$. An application of the Cauchy–Schwarz inequality to the sum

$$\sum_{\chi \neq \chi_0} P^{(k,k)} \left(\frac{L'}{L}(1, \chi) \right)$$

shows that for any $k \geq 1$,

$$\#\{\chi \neq \chi_0 : L'(1, \chi) \neq 0\} \geq \frac{T_{k,k}^2}{T_{2k,2k}}.$$

Let us note that

$$T_{k,k}^2 = (\mu^{(k,k)})^2 \varphi(q)^2 + O(\varphi(q) q^{\epsilon})$$

and that

$$\left(\mu^{(k,k)}\right)^2 = \sum_{n_1, n_2} \frac{\Lambda_k(n_1)^2 \Lambda_k(n_2)^2}{n_1^2 n_2^2} = \sum_{n=1}^{\infty} \frac{(\Lambda_k^2 \star \Lambda_k^2)(n)}{n^2},$$

where

$$(f \star g)(n) := \sum_{d|n} f(d) g(n/d),$$

is the Dirichlet convolution. Now, if $d(n)$ denotes the number of divisors of n,

$$\Lambda_{2k}(n)^2 = (\Lambda_k \star \Lambda_k)^2(n) = \left(\sum_{d|n} \Lambda_k(d) \Lambda_k(n/d) \right)^2 \leq d(n) \sum_{d|n} \Lambda_k^2(d) \Lambda_k^2(n/d)$$
$$= d(n) \left(\Lambda_k^2 \star \Lambda_k^2 \right)(n),$$

by an application of the Cauchy–Schwarz inequality. As $d(n) = O(n^{\epsilon})$ for any $\epsilon > 0$, we obtain

$$\mu^{(2k,2k)} = \sum_{n=1}^{\infty} \frac{\Lambda_{2k}^2(n)}{n^2} \leq \sum_{n=1}^{\infty} \frac{(\Lambda_k^2 \star \Lambda_k^2)(n)}{n^{2-\epsilon}}.$$

Putting

$$G_k(s) = \sum_{n=1}^{\infty} \frac{(\Lambda_k^2 \star \Lambda_k^2)(n)}{n^s}$$

we conclude

$$\frac{T_{k,k}^2}{T_{2k,2k}} \geq \frac{G_k(2)\varphi(q)^2 + O(\varphi(q)q^{\epsilon_2})}{G_k(2-\epsilon_1)\varphi(q) + O(q^{\epsilon_2})}$$

for any $\epsilon_1, \epsilon_2 > 0$. Choosing $k = 2$ and noting that

$$G_2(2-\epsilon_1) = G_2(2) + O(\epsilon_1),$$

we conclude that

$$\frac{T_{k,k}^2}{T_{2k,2k}} \geq \varphi(q) + O(q^{\epsilon}).$$

The result immediately follows from choosing $\epsilon_1 = 1/q$.

10 Concluding remarks

It is clear from the preceding discussion that the non-vanishing of certain Dirichlet series is connected with linear independence of special values of L-series. Such a theme was explored in a classical context in [9]. What is interesting in this paper is the role played by $L'(1,\chi)$ with χ a Dirichlet character.

The question of non-vanishing of $L'(1,\chi)$ arises in other contexts like the following. Let K be an algebraic number field and $\zeta_K(s)$ its Dedekind zeta function. It is well-known that $\zeta_K(s)$ has a simple pole at $s = 1$ with residue λ_K. Here,

$$\lambda_K = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w\sqrt{|d_K|}},$$

where r_1 is the number of real embeddings of K and $2r_2$ is the number of non-real embeddings of K, h_K, R_K, w and d_K are the class number, regulator, the number of units of finite order and discriminant, respectively, of K. Let us set

$$g_K(s) = \zeta_K(s) - \lambda_K\zeta(s).$$

Then, $g_K(s)$ is regular at $s = 1$. In [17], Scourfield asked if for any field $K \neq \mathbb{Q}$ we have $g_K(1) = 0$. This question is really about non-vanishing of linear combinations of derivatives of L-functions.

To see this, we write

$$\zeta_K(s) = \zeta(s)F_K(s),$$

where $F_K(s)$ is a product of certain Artin L-series. Using Brauer's induction theorem and the non-vanishing of Hecke L-series at $s = 1$, it is easily seen that $F_K(s)$ is regular at $s = 1$. Consequently, $F_K(1) = \lambda_K$ and since

$$\zeta_K(s) - \lambda_K \zeta(s) = \zeta(s)(F_K(s) - \lambda_K),$$

we see that $g_K(1) = F'_K(1)$. If $\widehat{K}$ denotes the normal closure of K over $\mathbb{Q}$, and $G = \mathrm{Gal}(\widehat{K}/\mathbb{Q})$, one can express $F_K(s)$ as a product of Artin L-series attached to irreducible characters of G. Indeed, if $H = \mathrm{Gal}(\widehat{K}/K)$, $\zeta_K(s)$ is the Artin L-series attached to the character $\mathrm{Ind}_H^G 1$. If χ is an irreducible character of G, we have by Frobenius reciprocity,

$$c_\chi := (\mathrm{Ind}_H^G 1, \chi) = (1, \chi|_H)$$

which is the multiplicity of the trivial character in χ restricted to H. Thus, c_χ is a non-negative integer and we have

$$F_K(s) = \prod_{\chi \neq 1} L(s, \chi)^{c_\chi},$$

where the product is over the non-trivial irreducible characters of G. Hence,

$$\frac{F'_K(1)}{F_K(1)} = \sum_{\chi \neq 1} c_\chi \frac{L'}{L}(1, \chi).$$

In the special case $K/\mathbb{Q}$ is Galois, $c_\chi = \chi(1)$. Thus, in the Galois case, the question of non-vanishing of $g_K(1)$ is equivalent to the non-vanishing of

$$\sum_{\chi \neq 1} \chi(1) \frac{L'}{L}(1, \chi).$$

If $K = \mathbb{Q}(\zeta_q)$ is the qth cyclotomic field, with ζ_q being a primitive qth root of unity, then Ihara et al. [10] have investigated the asymptotic behaviour of this sum. They proved that

$$\lim_{q \to \infty, q \text{ prime}} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_1} \frac{L'}{L}(1, \chi) = 0.$$

So the question of non-vanishing of $g_K(1)$ is a bit delicate and cannot be deduced from this limit theorem.

The non-vanishing of $L'(1, \chi)$ seems to be intimately linked with arithmetic questions. For example, if $K/\mathbb{Q}$ is quadratic, then $F_K(s) = L(s, \chi_D)$ where χ_D is the quadratic character attached to K. In this case, Scourfield's question reduces to the question of whether $L'(1, \chi_D) = 0$ for any such χ_D. It is unlikely that such a χ_D exists.

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