JOURNAL OF NUMBER THEORY 18, 241–248 (1984)

An Analogue of Artin's Conjecture for Abelian Extensions*

M. RAM MURTY

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Communicated by H. Stark

Received April 28, 1981

In 1927, Artin [1] enunciated the following hypothesis: given any nonzero integer $a \neq \pm 1$, or a perfect square, there exist infinitely many primes p for which a is a primitive root, modulo p. Moreover, if $N_a(x)$ denotes the number of such primes up to x, he conjectured that for a certain constant $A(a)$,

$$N_a(x) \sim A(a) \frac{x}{\log x},$$

as $x \to \infty$. In 1967, Hooley [3] proved this conjecture assuming the Riemann hypothesis for a certain (infinite) set of Dedekind zeta functions. Later, Goldstein [2] formulated a general conjecture, a special case of which was Artin's conjecture. His conjecture was as follows: for each prime q, let L_q be an algebraic number field, normal and of finite degree over $\mathbb{Q}$. For each squarefree k, set

$$L_k = \prod_{q \mid k} L_q,$$

where L_1 is taken to be $\mathbb{Q}$. Let $n(k) = [L_k : \mathbb{Q}]$. Then, the set of rational primes which do not split completely in any L_q has a natural density δ, where

$$\delta = \sum_{k=1}^{\infty} \frac{\mu(k)}{n(k)},$$

and μ denotes the usual Möbius function. Simple ideas from algebraic number theory reveal that Artin's conjecture is recaptured by the special case $L_q = \mathbb{Q}(\zeta_q, a^{1/q})$, where ζ_q is a primitive qth root of unity.

* Supported in part by NSF Grant MCS 77-18723 A03.

0022-314X/84 $3.00

Weinberger [7] showed that Goldstein's conjecture is not true, in general. His counterexample consisted of certain extensions L_q, abelian over $\mathbb{Q}$, and satisfying

$$q \ll \frac{\log |d_q|}{n(q)} \ll q,$$

where d_q denotes the discriminant of $L_q/\mathbb{Q}$. It was then realized that further conditions need to be imposed for the conjecture to be true.

Utilizing the methods of Hooley, Goldstein [2] proved the following theorem under the assumption of the generalized Riemann hypothesis.

THEOREM 1. *If for q sufficiently large, the extensions $L_q/\mathbb{Q}$ are abelian and*

(i) $1/n(k) \log |d_k| = O(\log k)$,

(ii) *if a prime p splits completely in L_q then for q sufficiently large, $p \geqslant f_q$, where f_q is the conductor of L_q,*

(iii) $\sum_{k>y} 1/n(k) = o(1/\log y)$, *as $y \to \infty$, then the set of primes which do not split completely in any L_q has a density*

$$\delta = \sum_{k=1}^{\infty} \frac{\mu(k)}{n(k)}.$$

The purpose of this paper is to supply an unconditional proof of this theorem. In fact, we shall prove a slightly general result from which the above theorem can be deduced.

Remarks. There seem to be numerous misprints in [2]. We address ourselves to [2] in these remarks. First, condition (i) of the theorem there should be as we have stated it above. Weinberger's counterexample confirms this. A careful study of the proof also reveals this fact (especially, Eq. 22, p. 109). The calculation in Eq. 24 should be

$$P(x, q) \leqslant t\left(\frac{x}{f_q} + 1\right) \leqslant \frac{x + \phi(f_q)}{n(q)}.$$

Therefore, we need to assume that if p splits completely in L_q, then $p \geqslant f_q$, *not* merely $p \geqslant q$. Finally, condition (iii) in (2) is insufficient to imply the absolute convergence of

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{n(k)}.$$

Counterexamples are easily constructed by taking an infinite tower for the L_q's.

To prove Theorem 1, we need the following lemmas.

LEMMA 1 (Lagarias–Odlyzko [4]). *Let $L/\mathbb{Q}$ be a normal extension of degree n and discriminant d (over $\mathbb{Q}$). There are effective constants A and B such that for*

$$x \geqslant \exp(10n(\log|d|)^2),$$

$$\pi(x, L) = \frac{\operatorname{li} x}{n} + \frac{\operatorname{li}(x^{\beta})}{n} + O(x \exp(-A\sqrt{\log x/n})),$$

where $\pi(x, L)$ denotes the number of rational primes $\leqslant x$ which split completely in L and $\operatorname{li} x$ denotes the familiar logarithmic integral, and if the β term is present at all then

$$\beta < 1 - \min\left(\frac{1}{4\log d}, \frac{1}{B|d|^{1/n}}\right).$$

The constant implied by the O symbol is absolute.

LEMMA 2. *With the same notation as in Lemma 1, there is an absolute constant c, such that if*

$$\sqrt{\log x/n} \geqslant c \max(\log|d|, |d|^{1/n}),$$

then

$$\pi(x, L) = \frac{\operatorname{li} x}{n} + O(x \exp(-A\sqrt{\log x/n})),$$

where the implied constant is absolute.

Proof. The lemma follows by utilizing the bound of Stark [6] for β given in Lemma 1.

The next lemma computes the discriminant of an arbitrary abelian extension $L/\mathbb{Q}$. Set for any natural number δ,

$$m(\delta) = [L \cap \mathbb{Q}(\zeta_\delta) : \mathbb{Q}],$$

where ζ_δ denotes a primitive δth root of unity. Let f be the conductor of L (i.e., the smallest f such that $L \subseteq \mathbb{Q}(\zeta_f)$).

LEMMA 3. *For an abelian extension $L/\mathbb{Q}$,*

$$\log |d| = m(f) \log f - \sum_{\delta \mid f} m\left(\frac{f}{\delta}\right) \Lambda(\delta),$$

where

$$\begin{aligned} \Lambda(n) &= \log p \qquad \text{if} \quad n = p^{\alpha},\ p \text{ prime}, \\ &= 0 \qquad\qquad \text{if not.} \end{aligned}$$

Proof. By the conductor discriminant formula,

$$|d| = \sum_{e \mid f} e^{s(e)},$$

where $s(e)$ is the number of characters of $\mathrm{Gal}(L/\mathbb{Q})$ which have conductor e. We know for any g,

$$\sum_{e \mid g} s(e) = m(g) = [L \cap \mathbb{Q}(\zeta_g) : \mathbb{Q}].$$

Möbius inversion gives

$$s(e) = \sum_{\delta \mid e} \mu\left(\frac{e}{\delta}\right) m(\delta).$$

Therefore,

$$\begin{aligned} \log |d| &= \sum_{e \mid f} s(e) \log e \\ &= \sum_{e \mid f} (\log e) \sum_{\delta \mid e} \mu\left(\frac{e}{\delta}\right) m(\delta) \\ &= \sum_{\delta \mid f} m(\delta) \sum_{d \mid e \mid f} \mu\left(\frac{e}{\delta}\right) \log e. \end{aligned}$$

The inner sum can be rewritten as

$$\sum_{t \mid (f/\delta)} \mu(t) \log(\delta t) = (\log \delta) \sum_{t \mid (f/\delta)} \mu(t) - \Lambda\left(\frac{f}{\delta}\right).$$

We finally get,

$$\log |d| = m(f) \log f - \sum_{\delta \mid f} m\left(\frac{f}{\delta}\right) \Lambda(\delta),$$

as desired.

COROLLARY 1. *For any abelian extension $L/\mathbb{Q}$ of degree n, discriminant d, and conductor f, we have*

$$\tfrac{1}{2}\log f \leqslant \frac{\log|d|}{n} \leqslant \log f.$$

Proof. As f is the conductor of L,

$$m\left(\frac{f}{\delta}\right) \leqslant \frac{m(f)}{2} \qquad \text{for} \quad \delta \neq 1.$$

By Lemma 3, we deduce

$$\log|d| \geqslant m(f)\log f - \frac{m(f)}{2}\sum_{\delta \mid f} \Lambda(\delta) \geqslant \frac{m(f)}{2}\log f = \frac{n}{2}\log f.$$

The inequality

$$\frac{1}{n}\log|d| \leqslant \log f$$

is easily deduced from the conductor-discriminant formula.

COROLLARY 2. *For any abelian extension $L/\mathbb{Q}$ of degree n and discriminant d,*

$$\frac{1}{n}\log|d| \geqslant \frac{1}{2}\log n.$$

LEMMA 4. *For any abelian extension $L/\mathbb{Q}$ of degree n and conductor f,*

$$\pi(x, L) \leqslant \frac{2x}{n\log(x/f)}.$$

Proof. There are reduced residue class representatives $a_1, \dots, a_t \pmod f$, where $t = \phi(f)/n$, such that any p splits completely in L if and only if one of $p \equiv a_1, \dots, p \equiv a_t \pmod f$ holds.

The Brun–Titchmarsh inequality [5] states that for $f < x$,

$$\pi(x, f, a) \leqslant \frac{2x}{\varphi(f)\log\left(\dfrac{x}{f}\right)},$$

where $\pi(x, f, a)$ is the number of primes $\leqslant x$ which are congruent to a $(\bmod f)$. Therefore,

$$\pi(x, L) = \sum_{i=1}^{t} \pi(x, f, a_i) \leqslant \frac{2tx}{\varphi(f) \log\left(\frac{x}{f}\right)} \leqslant \frac{2x}{n \log\left(\frac{x}{f}\right)},$$

as desired.

We now prove

THEOREM 2. *Suppose that for q sufficiently large, $L_q/\mathbb{Q}$ is abelian and that*

$$\sum_{k=1}^{\infty} \frac{\mu^2(k)}{n(k)} < \infty.$$

If

(i) $$\limsup_{k \to \infty} \frac{\log |d_k|}{n(k) \log k} < c < \infty, \textit{ and}$$

(ii) *for some* $0 < \theta < 1/2c$,

$$M(x^\theta) = o(x/\log x),$$

where $M(y)$ is the number of primes $\leqslant x$ which split completely in some L_q, $q > y$, then the set of primes which do not split completely in any L_q has a density

$$\delta = \sum_{k=1}^{\infty} \frac{\mu(k)}{n(k)}.$$

Proof. Let $N(x, y)$ be the number of primes $\leqslant x$ which do not split completely in any L_q, $q \leqslant y$. It is evident that for any y,

$$f(x) = N(x, y) + O(M(y)),$$

where $f(x)$ is the number of primes $\leqslant x$ which do not split completely in any L_q. Lemma 3 and condition (i) imply that for q sufficiently large,

$$\frac{1}{2} \log f_q \leqslant \frac{\log(d_q)}{n(q)} \leqslant c \log q,$$

where f_q denotes the conductor of $L_q/\mathbb{Q}$. Hence,

$$f_q \leqslant q^{2c}.$$

Moreover,

$$n(q) \leqslant \varphi(f_q) \leqslant q^{2c}.$$

Set $y = (1/16c) \log \log x$. Then

$$\prod_{q \leqslant y} q \leqslant (\log x)^{1/8c}.$$

Now, for squarefree $k \leqslant (\log x)^{1/8c}$, we have by (i),

$$n(k)\,|d_k|^{2/n(k)} \ll (\log x)^{1/4} \cdot (\log x)^{1/4} \ll (\log x)^{1/2}$$

and

$$n(k)(\log |d_k|)^2 \ll \log x.$$

This means, for $k \leqslant (\log x)^{1/8c}$, we can apply Lemma 2, and deduce

$$\begin{aligned} N(x, y) &= \sum_k{}' \mu(k)\, \pi(x, L_k) \\ &= \sum_k{}' \mu(k) \left\{ \frac{\operatorname{li} x}{n(k)} + O\left(x \exp\left(-A \sqrt{\frac{\log x}{n(k)}} \right) \right) \right\}, \end{aligned}$$

where the dash on the summation indicates that all prime divisors of k are $\leqslant y$. The error term is easily estimated by

$$\ll x(2^y) \exp(-A(\log x)^{3/8}) \ll x/(\log x)^2,$$

since $n(k) \leqslant (\log x)^{1/4}$. Therefore,

$$N(x, y) = \sum_k{}' \frac{\mu(k) \operatorname{li} x}{n(k)} + O\left(\frac{x}{(\log x)^2} \right).$$

The M term is handled easily. First,

$$M(y) \leqslant M(y, x^\theta) + M(x^\theta).$$

By (ii), $M(x^\theta) = o(x/\log x)$. By Lemma 4,

$$M(y, x^\theta) \leqslant \sum_{y < q < x^\theta} \frac{2x}{n(q) \log(x/f_q)}$$

and by (ii) $\theta < 1/2c$. As

$$f_q \leqslant q^{2c} \leqslant x^{2c\theta} \qquad \text{and} \qquad 2c\theta < 1,$$

we deduce

$$M(y, x)^{\theta} = o(x/\log x).$$

Therefore,

$$f(x) = \sum_k{}' \frac{\mu(k) \operatorname{li} x}{n(k)} + o(x/\log x)$$

and so

$$\lim_{x \to \infty} \frac{f(x)}{x/\log x} = \sum_{k=1}^{\infty} \frac{\mu(k)}{n(k)},$$

as desired.

We can deduce Theorem 1 from Theorem 2, since by (iii), and Lemma 4,

$$M(y) \leqslant \sum_{q \geqslant y} \frac{2x}{n(q)} = o(x/\log x)$$

for any $y = x^{\eta}$, $\eta > 0$.

Acknowledgment

It is a pleasure to thank Don Kersey for his help in the analysis of Lemma 3.

References

1. E. Artin, The collected papers of Emil Artin (S. Lang & J. T. Tate, Eds.), Addison–Wesley, Reading, Mass., 1965.
2. L. J. Goldstein, Some remarks on arithmetic density questions, *in* "Symposium in Pure Mathematics, St. Louis, 1972," Amer. Math. Soc., Providence, R. I.
3. C. Hooley, On Artin's conjecture, *J. Reine Angew. Math.* **225** (1967), 209–220.
4. J. Lagarias and A. Odlyzko, Effective versions of the Tchebotarev density theorem, *in* "Algebraic Number Fields" (A. Frölich, Ed.), Proceedings of the 1975 Durham Symposium, Academic Press, London/New York, 1977.
5. H. L. Montgomery and R. C. Vaughan, The large sieve, *Mathematika* **20** (1973), 119–134.
6. H. M. Stark, Some effective cases of the Brauer-Siegel theorem, *Invent. Math.* **23** (1974), 135–152.
7. P. J. Weinberger, A counterexample to an analogue of Artin's conjecture, *Proc. Amer. Math. Soc.* **35** (1972), 49–52.