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Inventiones mathematicae

An Ω-Theorem for Ramanujan's τ-Function

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§ 1. Introduction

Let $\tau(n)$ be defined by

$$\Delta(z)=\sum_{n=1}^{\infty}\tau(n)\,q^n=q\prod_{n=1}^{\infty}(1-q^n)^{24},\qquad q=e^{2\pi i z}.$$

This function was first studied by Ramanujan [6]. He wrote, for every prime p,

$$\tau(p)=2p^{11/2}\cos\theta_p$$

and conjectured that θ_p is real. This was proved by Deligne [2]. It is known that

$$\tau(p^\alpha)=p^{11\alpha/2}\,\frac{\sin(\alpha+1)\,\theta_p}{\sin\theta_p}.$$

If $d(n)$ denotes the number of divisors of n, then it follows that

$$|\tau(n)|\leqq n^{11/2}\,d(n),$$

as τ is a multiplicative function. Therefore, for some constant $c_1>0$,

$$\tau(n)=O\left(n^{11/2}\exp\left(\frac{c_1\log n}{\log\log n}\right)\right).$$

It is conjectured that

$$\tau(n)=\Omega\left(n^{11/2}\exp\left(\frac{c_2\log n}{\log\log n}\right)\right)\tag{1}$$

for some constant $c_2>0$.

A conjecture of Sato and Tate states that the angles θ_p are equidistributed in $[0,2\pi]$ with respect to the measure

$$\frac{2}{\pi}\sin^2\theta\,d\theta.$$

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It is easy to see that the conjecture of Sato-Tate implies (1). In fact, if

$$\operatorname{card}(p \leqq x\colon 0 \leqq \theta_p \leqq \varphi) \gg x^{\delta}$$

for some $\varphi < \frac{\pi}{3}$ and some $\delta > 0$, then (1) follows easily. Both assertions about the distribution of the angles θ_p remain unproved.

With respect to unconditional results, Rankin [5] showed

$$\limsup_{n \to \infty} \frac{|\tau(n)|}{n^{11/2}} = +\infty,$$

and Joris [3] proved

$$\tau(n) = \Omega(n^{11/2} \exp(c(\log n)^{(1/22)-\varepsilon})).$$

We shall show below that

$$\tau(n) = \Omega(n^{11/2} \exp(c(\log n)^{(2/3)-\varepsilon})).$$

For an arbitrary normalized Hecke eigenform

$$f = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

of weight k, a similar result is true if we assume that

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s}$$

has no real zeroes in the critical strip $k-1 \leqq \sigma \leqq k$.

Nevertheless, by an elementary method, one can show that

$$a_n = \Omega\left(n^{\frac{k-1}{2}} \exp\left(\frac{c(\log n)^{\frac{1}{k}}}{\log\log n}\right)\right).$$

This result remains true if f is a normalized eigenform of *even weight* for an arbitrary congruence subgroup of $SL_2(\mathbb{Z})$.

Notation. For the sake of brevity, we write

$$\tau_n = \tau(n)/n^{11/2},$$

and

$$f(s) = \sum_{n=1}^{\infty} \frac{\tau_n^2}{n^s}.$$

§ 2. Real Zeroes of $f(s)$

We show that $f(s)$ has no real zeroes in the critical strip $0 \leqq \sigma \leqq 1$.

Let $z = x + iy$ and set

$$\phi(z, s) = \frac{s(s-1)}{2} \left(\frac{y}{\pi}\right)^s \Gamma(s) \sum{}' |mz+n|^{-2s},$$

where the dash on the summation indicates we sum over all pairs of integers $(m, n) \neq (0,0)$. If we let

$$K(z,w)=\sum{}' \exp\left(-\frac{\pi w}{y}|mz+n|^2\right),$$

then it is easily seen that

$$\phi(z,s)=\frac{1}{2}+\frac{s(s-1)}{2}\int_1^\infty (w^{s-1}+w^{-s})\,K(z,w)\,dw \tag{2}$$

as

$$1+K(z,w)=\frac{1}{w}\left\{1+K\left(z,\frac{1}{w}\right)\right\}.$$

Letting

$$\psi(s)=(2\pi)^{-2(s+11)}\,\Gamma(s+11)\,\Gamma(s)\,\zeta(2s)\,f(s)\,s(s-1)$$

we see that

$$\psi(s)=\iint_{\mathscr{D}} y^{12}\cdot|\Delta(z)|^2\cdot\phi(z,s)\frac{dx\,dy}{y^2}, \tag{3}$$

where $\mathscr{D}$ denotes the standard fundamental domain for the full modular group acting on the upper half-plane. Also, ψ satisfies the functional equation.

$$\psi(s)=\psi(1-s).$$

In view of this functional equation and the fact that $f(s)$ has a simple pole at $s=1$, it suffices to consider $\frac{1}{2}\leqq s<1$ in our search for real zeroes.

Lemma 1.

$$\int_1^\infty K(z,w)\,dw\leqq\log\left(\frac{e^{\gamma+1}}{4\pi}\right)-2\log(y^{\frac{1}{2}}|\eta(z)|^2),$$

where γ is Euler's constant and

$$\eta(z)=e^{\pi iz/12}\prod_{n=1}^\infty(1-e^{2\pi inz}).$$

Proof. From Kronecker's limit formula (see e.g. Ramachandra [7]), it follows that

$$\lim_{s\to1}\left[\left(\frac{y}{\pi}\right)^s\Gamma(s)\sum{}'|mz+n|^{-2s}-\frac{1}{s-1}\right]=\log\left(\frac{e^\gamma}{4\pi}\right)-2\log(y^{\frac{1}{2}}|\eta(z)|^2).$$

But

$$\frac{2\phi(z,s)}{s(s-1)}=\frac{1}{(s-1)}-1+\int_1^\infty\left(1+\frac{1}{w}\right)K(z,w)\,dw+\text{higher powers of }(s-1)$$

so that

$$-1+\int_1^\infty\left(1+\frac{1}{w}\right)K(z,w)\,dw=\log\left(\frac{e^\gamma}{4\pi}\right)-2\log(y^{\frac{1}{2}}|\eta(z)|^2)$$

from which the result follows.

Corollary.

(i) *for* $\frac{\sqrt{3}}{2} \leqq y \leqq 2$, $\int_1^\infty K(z,w)\,dw \leqq \frac{1}{2}$,

(ii) *for* $y \geqq 1$, $\int_1^\infty K(z,w)\,dw \leqq \frac{\pi}{3} y$.

Proof. We have

$$\log|\eta(z)| = -\frac{\pi y}{12} + \sum_{n=1}^{\infty} \log|1-e^{2\pi i n z}|.$$

So that

$$-\log|\eta(z)| \leqq \frac{\pi y}{12} + \frac{e^{-2\pi y}}{(1-e^{-2\pi y})^2}. \tag{4}$$

It follows that

$$\int_1^\infty K(z,w)\,dw \leqq \frac{\pi}{3} y - \log y - 0.92$$

as

$$\log\left(\frac{e^{\gamma+1}}{4\pi}\right) = -0.95\ldots$$

and

$$\frac{e^{-2\pi y}}{(1-e^{-2\pi y})^2} \leqq 5.0 \times 10^{-3}$$

for $y \geqq \sqrt{3}/2$. Both (i) and (ii) are now easily deduced.

Theorem 1. $\psi(s) \neq 0$ *for* $\frac{1}{2} \leqq s \leqq 1$.

Proof. From (2) and (3), we observe that

$$\psi(s) = \frac{1}{2}(\Delta,\Delta) + \frac{s(s-1)}{2} \iint_{\mathscr{D}} y^{12}\,|\Delta(z)|^2 \left\{\int_1^\infty (w^{s-1}+w^{-s})\,K(z,w)\,dw\right\} \frac{dx\,dy}{y^2}$$

where $(\cdot,\cdot)$ denotes the Petersson inner product. It is apparent that for $\frac{1}{2} \leqq s \leqq 1$,

$$|\psi(s) - \tfrac{1}{2}(\Delta,\Delta)| \leqq \tfrac{1}{4} \iint_{\mathscr{D}} y^{12}\,|\Delta(z)|^2 \left(\int_1^\infty K(z,w)\,dw\right) \frac{dx\,dy}{y^2}.$$

By the corollary to Lemma 1, this is

$$\leqq \frac{1}{8}(\Delta,\Delta) + \frac{\pi}{12} \iint_{\substack{\mathscr{D} \\ y \geqq 2}} y^{11}\,|\Delta(z)|^2\,dx\,dy.$$

Noting that

$$\log|\Delta(z)| = 24 \log|\eta(z)| \leqq -2\pi y + 24 \cdot \frac{e^{-2\pi y}}{(1-e^{-2\pi y})^2},$$

we deduce

$$|\Delta(z)| \leqq (1 \cdot 1)\, e^{-2\pi y}$$

for $y \geqq \sqrt{3}/2$.

This estimate implies

$$\iint_{\substack{\mathscr{D}\\ y\geqq 2}} y^{11}|\Delta(z)|^2\,dx\,dy \leqq (1\cdot 21)\int_2^\infty y^{11}e^{-4\pi y}\,dy \leqq (1\cdot 21)\left(\frac{2}{e}\right)^{4\pi}\frac{e^{-4\pi}}{4\pi}$$

$$\leqq (0\cdot 04)\frac{e^{-4\pi}}{4\pi}.$$

It follows that

$$\psi(s) \geqq \frac{3}{8}(\Delta,\Delta) - (0.01)\frac{e^{-4\pi}}{4\pi}.$$

We note that, if $F(x)=\sum c_{ne}^{2\pi i n z}$, then for $k\geqq 2$,

$$\iint_{\substack{|x|<\frac{1}{2}\\ y>1}} y^k|F(z)|^2\frac{dx\,dy}{y^2} = \sum_{n=1}^\infty |c_n|^2\int_1^\infty y^{k-2}e^{-4\pi n y}\,dy$$

$$\geqq |c_1|^2\int_1^\infty e^{-4\pi y}\,dy$$

$$= |c_1|^2\frac{e^{-4\pi}}{4\pi}.$$

Taking in particular, $k=12$, $c_n=\tau(n)$, we have

$$(\Delta,\Delta) \geqq \frac{e^{-4\pi}}{4\pi}.$$

We finally obtain

$$\psi(s) \geqq \frac{e^{-4\pi}}{16\pi} > 0$$

for $\frac{1}{2}\leqq s\leqq 1$.

Remarks. 1. Lehmer [4] has computed $(\Delta,\Delta)=1.036\times 10^{-6}$.

2. It is possible to estimate

$$\int_1^\infty K(z,w)\,dw$$

without appealing to Kronecker's limit formula. We split the sum

$$\sum{}' \int_1^\infty \exp\left(-\frac{\pi w}{y}|mz+n|^2\right)dw$$

into fours parts corresponding to $n=0$, $m=0$, $|n|\leqq|m|\,y$ and $|n|>|m|\,y$, where in the latter two cases, we utilise the inequalies $|mz+n|^2\geqq m^2y^2$ and $|mz+n|^2\geqq\frac{3}{4}|n|^2$ in the respective cases. The resulting four sums are easily estimated and the main contribution arises from the term corresponding to $n=0$.

We indicate another proof of Theorem 1 which can be based on the following idea. From Chowla-Selberg [1, p. 106] we know

$$\frac{2\phi(z,s)}{s(s-1)} = \frac{\zeta(2s)\,y^s}{s(2s-1)} + \frac{\zeta(2s-1)\,y^{1-s}}{(s-1)(2s-1)} + R(y,s)$$

where

$$\xi(s)=\frac{s(s-1)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

and

$$|R(y,s)|\leqq\frac{8}{\pi\sqrt{y}}\cdot\frac{1}{e^{\pi y}-1}$$

for $\frac{1}{2}\leqq s\leqq 1$.

A simple calculation reveals $|R(y,s)|\leqq 0.01$. Utilizing the fact that

$$\xi(s)=\frac{1}{2}+\frac{s(s-1)}{2}\int_1^\infty \psi(x)(x^{s-1}+x^{-s})\,dx,$$

where

$$\psi(x)=\sum_1^\infty e^{-n^2\pi x},$$

it is straightforward to show that for $y\leqq 2$.

$$\frac{2\phi(z,s)}{s(s-1)}\leqq\frac{1}{2s-1}\left(\frac{y^s}{s}-\frac{y^{1-s}}{1-s}\right)+0.15.$$

A simple application of Rolle's theorem reveals that

$$\frac{2\phi(z,s)}{s(s-1)}\leqq -0.1$$

for $y\leqq 2$.

A similar argument shows that for $y\geqq 2$, and $\frac{1}{2}<s\leqq 1$,

$$\frac{2\phi(z,s)}{s(s-1)}\leqq y^2.$$

These two inequalities are enough for Theorem 2 to be deduced.

§ 3. Zeroes of $\psi(s)$ in the Critical Strip

Let $N(T,\psi)$ be the number of zeroes of $\psi(s)$ satisfying $0<\sigma<1$ and $0<t<T$.

Lemma 2.

$$N(T,\psi)=\frac{2}{\pi}T\log T+O(T).$$

Proof. Let R be the rectangle with vertices $\frac{3}{2}$, $\frac{3}{2}+iT$, $-\frac{1}{2}+iT$, $-\frac{1}{2}$. In view of the functional equation and the fact that $\psi(s)$ has no real zeroes in $-\frac{1}{2}\leqq\sigma\leqq\frac{3}{2}$, we see that

$$\pi N(T,\psi)=\Delta_L \arg\psi(s),$$

where Δ_L denotes the variation in the argument as s traverses from $\frac{3}{2}$ to $\frac{3}{2}+iT$ and then to $\frac{1}{2}+iT$. Stirling's formula easily gives

$$\Delta_L \arg((2\pi)^{-2s-22}s(s-1)\Gamma(s)\Gamma(s+11))=2T\log T+O(T).$$

Moreover, as $\psi(s)$ is of order 1, it is deduced, in a standard way, that

$$\sum_{\rho} \frac{1}{1+(T-\gamma)^2} = O(\log T),$$

as ρ runs through the zeroes of $\psi(s)$.

It follows that the number of zeroes $\sigma+i\gamma$, with $|T-\gamma|<1$ is $O(\log T)$ and

$$\frac{2\zeta'}{\zeta}(2s)+\frac{f'}{f}(s)=\sum_{\rho}{}' \frac{1}{(s-\rho)}+O(\log t),$$

where the dash on the summation indicates the sum is over zeroes of $\psi(s)$ for which $|t-\gamma|<1$, $\rho=\sigma+i\gamma$. We have

$$\begin{aligned}\Delta_L \arg(\zeta(2s)\, f(s)) &= \int_L \operatorname{Im}\left(2\frac{\zeta'}{\zeta}(2s)+\frac{f'}{f}(s)\right) ds \\ &= O(1) - \int_{\frac{1}{2}+iT}^{\frac{3}{2}+iT} \operatorname{Im}\left(2\frac{\zeta'}{\zeta}(2s)+\frac{f'}{f}(s)\right) ds,\end{aligned}$$

the $O(1)$ term coming from the variation along $\sigma=\frac{3}{2}$. As

$$\int_{\frac{1}{2}+iT}^{\frac{3}{2}+iT} \operatorname{Im}(s-\rho)^{-1}\, ds = \Delta \arg(s-\rho) = O(1)$$

for those zeroes ρ satisfying $|t-\gamma|<1$, we deduce

$$\Delta_L(\arg(\zeta(2s)\, f(s))) = O(\log T).$$

This completes the proof.

§4. Other Lemmas

Lemma 3. *Let $\tau_p^2>1$. For such a prime p, there is an $m=m(p)$ and an absolute constant c such that $\tau_{p^m}\geqq c>1$ and*

$$m(p) \ll \frac{1}{\tau_p^2-1}.$$

Proof. If $\tau_p^2-1>10^{-10}$, then take $m(p)=1$. Now suppose

$$0<\tau_p^2-1<10^{-10}.$$

Then θ_p is close to $\frac{\pi}{3}$ or $\frac{2\pi}{3}$, we consider the case θ_p close to $\frac{\pi}{3}$, the other case being similar. Also, $0<\theta_p<\frac{\pi}{3}$. If $\theta_p<\frac{\pi}{6}$, we may take $m(p)=1$. So we may assume $\frac{\pi}{6}<\theta_p<\frac{\pi}{3}$.

Choose $m\equiv 0 \pmod 6$ such that

$$\frac{\frac{\pi}{10}}{\frac{\pi}{3}-\theta_p}<m+1<\frac{\frac{\pi}{10}}{\frac{\pi}{3}-\theta_p}+20$$

so that $\sin(m+1)\,\theta_p=\sin\left((m+1)\frac{\pi}{3}+(m+1)\left(\theta_p-\frac{\pi}{3}\right)\right)\geqq\sin\left(\frac{\pi}{3}+\frac{\pi}{10}\right)$, as

$$\left|\theta_p-\frac{\pi}{3}\right|<2\,|\sin\theta_p|\left|\theta_p-\frac{\pi}{3}\right|\leqq|2\cos\theta_p-1|\leqq\tau_p^2-1\leqq 10^{-10}.$$

Therefore,

$$\frac{\sin(m+1)\,\theta_p}{\sin\theta_p}\geqq\frac{\sin\left(\frac{\pi}{3}+\frac{\pi}{10}\right)}{\sin\frac{\pi}{3}}>1.$$

Moreover, m satisfies

$$m\ll\frac{1}{\frac{\pi}{3}-\theta_p}\leqq\frac{\sqrt{3}}{\tau_p-1}\leqq\frac{3\sqrt{3}}{\tau_p^2-1}.$$

This completes the proof.

Lemma 4.

$$\sum_{\substack{p\\ \tau_p^2>1}}\frac{\tau_p^2-1}{p^{\beta}}=+\infty,$$

for $\beta<\frac{1}{2}$.

Proof. Set $\theta(s)=\frac{\zeta(2s)}{\zeta(s)}f(s)$. We know

$$\log\theta(s)=\sum_{p,n}\frac{2\cos n\,\theta_p+1}{n p^{ns}}=\sum_p\frac{\tau_p^2-1}{p^s}\left(1+\frac{\tau_p^2-3}{2p^s}+\dots\right).$$

Now write

$$\sum_p\frac{\tau_p^2-1}{p^s}=f_+(s)-f_-(s)$$

where

$$f_+(s)=\sum_{\tau_p^2>1}\frac{\tau_p^2-1}{p^s}$$

and

$$f_-(s)=-\sum_{\tau_p^2<1}\frac{\tau_p^2-1}{p^s}.$$

Suppose that $f_+(\frac{1}{2})<\infty$. Then, for $\sigma>\frac{1}{2}$, $f_+(s)$ is analytic. By Lemma 2, $\log\theta(s)$ has singularities with $\operatorname{Re}s\geqq\frac{1}{2}$ arising from the zeroes of $\psi(s)$. The set of singularities of $\log\theta(s)$ coincides with the set of singularities of $f_-(s)$ for $\mathrm{Re}\, s>\frac{1}{2}$. If this set is not empty, $f_-(s)$ has a real singularity by Landau's theorem. Therefore, $\psi(s)$ has a real zero which contradicts Theorem 1. Therefore, all the singularities of $\log\theta(s)$ lie on the line $\sigma=\frac{1}{2}$. As $\log\theta(s)$ is analytic at $s=\frac{1}{2}$, both $f_+(s)$ and $f_-(s)$ have a singularity at $s=\frac{1}{2}$. Therefore,

$$\sum_{\substack{p \\ \tau_p^2>1}} \frac{\tau_p^2-1}{p^{\frac{1}{2}-\varepsilon}}=+\infty$$

and

$$\sum_{\substack{p \\ \tau_p^2<1}} \frac{\tau_p^2-1}{p^{\frac{1}{2}-\varepsilon}}=-\infty.$$

This completes the proof of the lemma.

§5. Main Theorem

Theorem 2. *Suppose* $\sum_{\tau_p^2>1} \frac{\tau_p^2-1}{p^\beta}=+\infty$. *Then*

$$\tau_n=\Omega(\exp(c(\log n)^{\frac{1}{2-\beta}-\varepsilon})).$$

Proof. Clearly, the set

$$S=\left\{m: \sum_{\substack{e^m<p<e^{m+1} \\ \tau_p^2>1}} \frac{\tau_p^2-1}{p^\beta} \geqq \frac{2}{m^2}\right\}$$

is infinite. Since

$$\sum_{\substack{e^m<p<e^{m+1} \\ 0<\tau_p^2-1<\frac{1}{p^{1-\beta}\log p}}} \frac{\tau_p^2-1}{p^\beta} \leqq \int_{e^m}^{e^{m+1}} \frac{dt}{t\log^2 t} \leqq \frac{1}{m^2},$$

we see that

$$T=\left\{m: \sum_{\substack{e^m<p<e^{m+1} \\ 3>\tau_p^2-1>\frac{1}{p^{1-\beta}\log p}}} \frac{\tau_p^2-1}{p^\beta} \geqq \frac{1}{m^2}\right\}$$

is infinite.

For each $m\in T$, we know

$$\sum_{\frac{1}{100}<\tau_p^2-1<3} + \sum_{p^{-\varepsilon}<\tau_p^2-1<\frac{1}{100}} +\dots+ \sum_{\frac{p^{\beta-1}}{\log p}<\tau_p^2-1<p^{\beta-1+\varepsilon}} \frac{\tau_p^2-1}{p^\beta} \geqq \frac{1}{m^2}.$$

The number of sums is $O_\varepsilon(1)$. Therefore, for some γ,

$$\sum_{p^{-\gamma-\varepsilon}<\tau_p^2-1<p^{-\gamma}} \frac{\tau_p^2-1}{p^\beta} \geqq \frac{c(\varepsilon)}{m^2}.$$

Hence if

$$W=\{p\colon e^m<p<e^{m+1}, p^{-\gamma-\varepsilon}<\tau_p^2-1<p^{-\gamma}\},$$

then

$$|W|\, e^{-m(\beta+\gamma)} \geqq \frac{c(\varepsilon)}{m^2}.$$

Now, set $B \subseteq W$ such that $|B| = \frac{(e^m)^{\beta+\gamma}}{m^2}$, and define

$$n = \prod_{\substack{p \leqq x \\ p \in B}} p^{m(p)}$$

where $m(p)$ is defined by Lemma 3. Then by Lemma 3,

$$\tau_n \geqq \exp\left(\frac{c x^{\beta+\gamma}}{(\log x)^2}\right).$$

Since,

$$\log n \leqq \sum_{\substack{p \leqq x \\ p \in B}} p^{\gamma+\varepsilon} \log p \leqq \frac{x^{\beta+2\gamma+\varepsilon}}{(\log x)}$$

we have

$$x^{\beta+2\gamma+\varepsilon} \geqq (\log n)(\log x).$$

As

$$\log n \geqq \sum_{\substack{p \leqq x \\ p \in B}} \log p \geqq \frac{x^{\beta+\gamma}}{(\log x)^2},$$

we have

$$\tau_n \geqq \exp\left(\frac{c((\log n)(\log\log n))^{\frac{\beta+\gamma}{\beta+2\gamma+\varepsilon}}}{(\log\log n)^2}\right).$$

Noting that,

$$0 \leqq \gamma+\varepsilon \leqq 1-\beta,$$

we finally deduce

$$\tau_n \geqq \exp(c(\log n)^{\frac{1}{2-\beta}-\varepsilon}),$$

as desired.

Corollary 1. $\tau_n = \Omega(\exp(c(\log n)^{\frac{2}{3}-\varepsilon}))$.

Proof. By Lemma 4 any $\beta < \frac{1}{2}$ satisfies the condition of the theorem. This gives the result.

Remarks. 1. By utilizing the fact that

$$\sum_{\tau_p^2<1} \frac{\tau_p^2-1}{p^{\frac{1}{2}-\varepsilon}} = -\infty$$

and repeating the above argument, one can deduce that

$$|\tau_n| < \exp(-c(\log n)^{\frac{2}{3}-\varepsilon})$$

for an infinity of n.

2. The argument can be extended to any *real valued* multiplicative function c satisfying

(i) $c(p)^2-1=c(p^2)$

(ii) $c(n)=O(n^{\varepsilon})$

(iii) the Dirichlet series $\frac{\zeta(2s)}{\zeta(s)}\sum_{n=1}^{\infty}\frac{c(n)^2}{n^s}=L(s)$ (say) has an analytic continuation to $\operatorname{Re} s=0$.

Also, if $L(s)$ has only non-real zeroes in $\operatorname{Re} s\geqq\frac{1}{2}$, then

$$c(n)=\Omega(\exp(c(\log n)^{\frac{1}{2}-\varepsilon})).$$

In fact, if $\operatorname{Re} s\geqq\sigma$ is the largest zero-free half-plane for $L(s)$, then

$$c(n)=\Omega(\exp(c(\log n)^{\sigma-\varepsilon})).$$

§ 6. General Results

Let $f(z)=\sum_{n=1}^{\infty}a(n)\,e^{2\pi inz}$ be a normalized Hecke eigenform of weight k for the full modular group. Then k is even and as the $a(n)$'s are integers, we have for primes p,

$$|a(p)^2-p^{k-1}|\geqq 1.$$

If we let

$$a_p=a(p)\Big/p^{\frac{k-1}{2}},$$

then

$$|a_p^2-1|\geqq\frac{1}{p^{k-1}}.$$

Lemma 5. *There is an $m=m(p)$ such that $|a_{p^m}|\geqq c>1$, where c is a fixed constant and*

$$m(p)\ll\frac{1}{|a_p^2-1|}.$$

The proof of this lemma proceeds exactly as in Lemma 3 and therefore, we suppress it.

By the preceding remarks, it is evident that $m(p)$ in Lemma 5 satisfies

$$m(p)\ll p^{k-1}.$$

Therefore, if we let

$$n=\prod_{p\leqq x}p^{m(p)},$$

then

$$a_n\geqq c^{\pi(x)}.$$

But

$$\log n=\sum_{p\leqq x}m(p)\log p\leqq x^k.$$

This proves the following theorem.

Theorem 3. *If* $f(z)=\sum_{1}^{\infty} a(n)\, e^{2\pi i n z}$ *is a normalized Hecke eigenform, weight k, then*

$$a(n)=\Omega\left(n^{\frac{k-1}{2}} \exp\left(\frac{c(\log n)^{\frac{1}{k}}}{\log\log n}\right)\right).$$

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