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Smooth arithmetical sums over k-free integers

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Abstract. We use partial zeta functions to analyse the asymptotic behaviour of certain smooth arithmetical sums over smooth k-free integers.

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1. Introduction

Let $\Omega(n)$ be the number of prime divisors of n counted with multiplicity. Let $f : \mathbb{R} \to \mathbb{C}$ be a bounded function, and $z \in \mathbb{C}$. Fix an integer $k \geq 2$. We say that an integer n is *k-free* if its prime factors are all of multiplicity less than k. We want to study the sum

$$S_{\Omega,f}(z,k;N) := \sum_{\substack{n \text{ is } k-\text{free} \\ p|n \Rightarrow p \leq N}} f\left(\frac{\log n}{\log N}\right) \frac{z^{\Omega(n)}}{n} \tag{1}$$

as $N \to \infty$. It is customary to refer to the integers n in the above sum as *smooth* (or *friable*) due to the constraint on the size of their prime divisors. Such sums have been studied by Cellarosi [2] and Avdeeva, Li, and Sinai [1] using a combination of ideas from analytic number theory and statistical mechanics. Here, our goal is to use only methods from analytic number theory – specifically partial zeta functions – to derive similar results. Our method is a variation of a similar technique used by Murty and Vatwani [10] in their work on the higher rank Selberg sieve. We prove the following

Theorem 1.1. *Suppose that f is of Schwartz class. Then for every $\theta > 1$ we have, as $N \to \infty$,*

$$S_{\Omega,f}(z,k;N) = C_f(z,k;N)\,(\log N)^z \left(1 + O_\theta(\log^{-\theta} N)\right),$$

where $C_f(z,k;N)$ has an explicit expression (see (35) *below) and is $O(1)$.*

Theorem 1.1 follows from a more general result (see Theorem 8.1 below) in which finite regularity for f is assumed.

For context and motivation for the study of the sums (1) we refer the reader to Section 2 of [2]. A lot of work on arithmetic and logarithmic averages of multiplicative functions over smooth integers has been done by Tenenbaum and Wu [14–16] and Hanrot, Tenenbaum and Wu [5]. They consider sums of the form

$$\Psi_g(x,y) = \sum_{n \in S(x,y)} g(n), \tag{2}$$

$$\psi_g(x,y) = \sum_{n \in S(x,y)} \frac{g(n)}{n}, \tag{3}$$

where $S(x,y)=\{1\le n\le x:\ p|n\Rightarrow p\le y\}$. Many authors assume some constraints on the sizes of x and y, usually expressed in terms of $u=\frac{\log x}{\log y}$. For instance, Song [11] studies sums of the form (2) where $1\le u\le\frac{(\log y)^{\delta/2}}{\log\log y}$ and the power $0<\delta<1$ satisfies $\sum_{p\le x} g(p)\log p=\kappa x+O(x/(\log x)^{\delta})$ for some $\kappa>1$. Song's result has been extended by Tenenbaum and Wu [14] to the wider range $u\log(u+1)\ll(\log y)^{(\delta+\delta_1)/2}$, where $\delta_1=\min\{\delta,1\}$ and $\kappa,\delta>0$ satisfy the condition $\kappa+\delta\ge 1$. The aforementioned works all assume that g is a *positive* multiplicative function satisfying some additional conditions on its average along primes and prime powers, see e.g. [14]. For the case of a constant function g, see (14)–(15) below.

To see the analogy between (1) and (3), observe that $n\mapsto z^{\Omega(n)}$ is a (complex-valued) multiplicative function and hence (1) can be seen as a weighted version of (3), restricted to k-free integers. If the support of the weight function f contains $\mathbb{R}_{\ge 0}$, then our assumption that n is k-free and the restriction that the largest prime dividing n is at most N imply that $n\le(p_1\cdots p_{\pi(N)})^{k-1}=e^{(k-1)N(1+o(1))}$ as $N\to\infty$. Therefore we are in the case where $u\sim(k-1)\frac{N}{\log N}$, which is larger than what is allowed in [14]. However, we can only handle such a large range for u because of the weight f, and hence our work does not generalize [11] or [14].

If we could take $f=\mathbf{1}_{[0,u]}$ and drop the k-free condition, then we would obtain $\psi_g(N^u,N)$ with $g(n)=z^{\Omega(n)}$ and the asymptotic behaviour for the corresponding sum would follow from Theorem 2.1 in [14], at least when $z\in\mathbb{R}^+$. Our approach, however, does not allow such a sharp weight function f since its Fourier transform does not decay fast enough (see (5) and (33)). Nonetheless, we are able to obtain asymptotic results for *arbitrary* $z\in\mathbb{C}$ provided the weight function f is sufficiently regular (the amount of regularity needed increases the more negative the real part of z is, see Theorem 8.1). For the study of sums of the form (2) with $g(n)=z^{\Omega(n)}$ and $|z|=1$, see also Theorem 2 in [8]. Our assumptions of z and f being quite general, it is hard to describe explicitly the asymptotic behaviour of the quantity $C_f(k,z;N)$ as $N\to\infty$.

The paper is organized as follows. In Sections 2. and 3. we rewrite the sum $S_{\Omega,f}(z,k;N)$ in terms of the inverse Fourier transform of f and a partial zeta function. In Sections 4. and 5. we gather some results concerning the Dickman function and partial zeta functions. In Sections 6. and 7. we isolate the main term and estimate all the error terms in our analysis. The more general version of Theorem 1.1 is presented in Section 8.. Some remarks on how to find the limit of $C_f(k,z;N)$ as $N\to\infty$ and a concrete example are provided in Section 9..

2. Rewriting the sum via Fourier transform

Writing f as a Fourier transform, we have

$$f(t)=\int_{-\infty}^{\infty}\widehat{f}(x)e^{-ixt}\,dx,$$

and we obtain that $S_{\Omega,f}(z,k;N)$ is equal to

$$\sum_{\substack{n\text{ is }k-\text{free}\\ p|n\Rightarrow p\le N}}\left(\int_{-\infty}^{\infty}\widehat{f}(x)e^{-ix\frac{\log n}{\log N}}dx\right)\frac{z^{\Omega(n)}}{n}=\int_{-\infty}^{\infty}\widehat{f}(x)\left(\sum_{\substack{n\text{ is }k-\text{free}\\ p|n\Rightarrow p\le N}}\frac{z^{\Omega(n)}}{n}n^{-\frac{ix}{\log N}}\right)dx \tag{4}$$

under suitable regularity conditions on f. More precisely, we will assume that $\widehat{f}$ satisfies the bound

$$|\widehat{f}(x)|\ll\frac{1}{(1+x^2)^{\frac{\eta}{2}}} \tag{5}$$

for suitably large $\eta>0$. If f is assumed to be of Schwartz class, then (5) holds for every $\eta>0$.

3. Partial zeta functions

We aim to rewrite the integrand in the right-hand-side of (4) using partial zeta functions.

$$\zeta(s, N) := \prod_{p \leq N} \left(1 - \frac{1}{p^s}\right)^{-1}$$

(see, e.g. Section III.5 of [13]). The function

$$g_{z,k,N}(s) := \sum_{\substack{n \text{ is } k\text{-free} \\ p|n \Rightarrow p \leq N}} \frac{z^{\Omega(n)}}{n^s} = \prod_{p \leq N} \left(1 + \frac{z}{p^s} + \frac{z^2}{p^{2s}} + \cdots + \frac{z^{k-1}}{p^{(k-1)s}}\right) \tag{6}$$

can easily be simplified since

$$1 + \frac{z}{p^s} + \frac{z^2}{p^{2s}} + \cdots + \frac{z^{k-1}}{p^{(k-1)s}} = \frac{1 - \frac{z^k}{p^{ks}}}{1 - \frac{z}{p^s}}.$$

Thus

$$g_{z,k,N}(s) = \prod_{p \leq N} \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{z^k}{p^{ks}}\right).$$

By the binomial theorem, we can rewrite

$$\left(1 - \frac{z}{p^s}\right)^{-1} = \left(1 - \frac{1}{p^s}\right)^{-z} \left(1 + \frac{z^2 - z}{2p^{2s}} + \frac{z^3 - z}{3p^{3s}} + \frac{3z^4 - 2z^3 + z^2 - 2z}{8p^{4s}} + \cdots\right)$$

so that

$$g_{z,k,N}(s) = \prod_{p \leq N} \left(1 - \frac{1}{p^s}\right)^{-z} h_{z,k,N}(s), \tag{7}$$

where

$$h_{z,k,N}(s) := \prod_{p \leq N} \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{z} \left(1 - \frac{z^k}{p^{ks}}\right) \tag{8}$$

$$= \prod_{p \leq N} \left(1 + \frac{z^2 - z}{2p^{2s}} + \frac{z^3 - z}{3p^{3s}} + \frac{3z^4 - 2z^3 + z^2 - 2z}{8p^{4s}} + \cdots\right)\left(1 - \frac{z^k}{p^{ks}}\right) \tag{9}$$

is a bounded function near $s = 1$ since $k \geq 2$. We recognize then that

$$g_{z,k,N}(s) = \zeta(s, N)^z \, h_{z,k,N}(s)$$

with $h_{z,k,N}(s)$ actually uniformly bounded for $N \geq 1$ and s in the half-plane $\Re(s) > \frac{3}{4}$.

Combining (4), (6), and (7), we can rewrite $S_{\Omega,f}(z, k; N)$ as

$$\int_{-\infty}^{\infty} \widehat{f}(x) \, g_{z,k,N}\left(1 + \frac{ix}{\log N}\right) dx = \int_{-\infty}^{\infty} \widehat{f}(x) \, \zeta\left(1 + \frac{ix}{\log N}, N\right)^z \, h_{z,k,N}\left(1 + \frac{ix}{\log N}\right) dx. \tag{10}$$

As we will see, the main contribution to the sum $S_{\Omega,f}(z, k; N)$ will come from the integral in (10) where $|x| \leq 3 \log N$. If $\tau = \frac{x}{\log N}$, then we will be required to integrate a product of various functions, including $h_{z,k,N}(s)$, where $s = 1 + i\tau$ and $|\tau| \leq 3$. In this region, we claim that the function $h_{z,k,N}(s)$ is very close to the function

$$h_{z,k}(s) := \prod_{p} \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{z} \left(1 - \frac{z^k}{p^{ks}}\right). \tag{11}$$

Lemma 3.1. *As $N \to \infty$ we have, uniformly in $|\tau| \le 3$,*

$$h_{z,k,N}(1+i\tau) = h_{z,k}(1+i\tau)\left(1 + O\left(\frac{1}{N\log N}\right)\right). \tag{12}$$

Proof. Using (9), the relative difference $(h_{z,k,N}(s) - h_{z,k}(s))/h_{z,k,N}(s)$ can be expressed as

$$1 + \exp\left\{\sum_{p>N}\log\left(1 + \frac{z^2-z}{2p^{2s}} + \frac{z^3-z}{3p^{3s}} + \cdots\right) + \sum_{p>N}\log\left(1 - \frac{z^k}{p^{ks}}\right)\right\}. \tag{13}$$

Note that for $s = 1+i\tau$ the two sums in (13) are $O\big(\sum_{p>N}\frac{1}{p^2}\big)$ and $O\big(\sum_{p>N}\frac{1}{p^k}\big)$, respectively, as $N \to \infty$. Moreover, since $k \ge 2$, the first sum dominates the second. Finally, we use integration by parts:

$$\sum_{p>N}\frac{1}{p^2} = \int_N^\infty \frac{1}{x^2}\,d\pi(x) = \left[\frac{\pi(x)}{x^2}\right]_N^\infty + 2\int_N^\infty \frac{\pi(x)}{x^3}\,dx \ll \frac{1}{N\log N}.$$

□

4. On the Dickman function

The Dickman function ρ is defined as the solution to the delay differential equation $u\rho'(u) + \rho(u-1) = 0$ for $u > 1$ and $\rho(u) = 1$ for $0 < u \le 1$. It appears naturally when counting smooth integers. Namely, if we define

$$\Psi(x,y) = \sum_{\substack{n\le x\\ p|n \Rightarrow p\le y}} 1, \tag{14}$$

then Dickman [3] proved that $\Psi(x,y) \sim x\rho(u)$ as $x \to \infty$ when $y = x^{1/u}$ and $u \ge 1$ is fixed. This asymptotic result has been refined and extended to values of u that may depend on x. For instance, Hildebrand [7] proved that

$$\Psi(x,y) = x\rho(u)\left(1 + O_\varepsilon\left(\frac{\log(u+1)}{\log y}\right)\right), \quad \text{where} \quad y = x^{1/u} \tag{15}$$

for $1 \le u \le \exp\{(\log y)^{3/5-\varepsilon}\}$. It is also known that (15) holds uniformly for $1 \le u \le y^{1/2-\varepsilon}$ if and only if the Riemann Hypothesis is true (Hildebrand [6]). The only property of the Dickman function we shall use is that its Laplace transform $\widehat{\rho}$ satisfies

$$s\widehat{\rho}(s) = e^{-J(s)}, \quad J(s) = \int_0^\infty \frac{e^{-s-t}}{s+t}dt, \tag{16}$$

and $s \mapsto J(s)$ is holomorphic on $\mathbb{C}\setminus(-\infty,0]$, see e.g. section III.5.4 of [13]. In particular, by studying the function $x \mapsto \exp\big(-\int_0^\infty \frac{e^{-ix-t}}{ix+t}dt\big)$, it is not hard to show that there are two constants $C_1, C_2 > 0$ such that

$$\frac{C_1}{(1+x^2)^{\frac{1}{2}}} \le |\widehat{\rho}(ix)| \le \frac{C_2}{(1+x^2)^{\frac{1}{2}}}$$

for every $x \in \mathbb{R}$. Therefore the bound

$$\left|\widehat{\rho}(ix)^z\right| \ll \frac{1}{(1+x^2)^{\frac{\Re(z)}{2}}} \tag{17}$$

holds for every $z \in \mathbb{C}$, uniformly in x.

5. Tenenbaum's Lemma for partial zeta functions

The function $\zeta(\cdot, N)$ has been studied extensively. For instance, we have the following

Lemma 5.1 (Tenenbaum. See Lemma 9.1 on page 378 of [12]). *Let $\varepsilon > 0$. Then there exists $N_0 = N_0(\varepsilon) \geq 0$ such that, under the conditions*

$$N \geq N_0, \quad \sigma \geq 1 - (\log N)^{-(2/5)-\varepsilon}, \quad |\tau| \leq L_\varepsilon(N) := \exp\left\{(\log N)^{3/5-\varepsilon}\right\}, \tag{18}$$

we have uniformly

$$\zeta(s, N) = \zeta(s)(s-1)(\log N)\,\widehat{\rho}\,((s-1)\log N)\left\{1 + O\left(\frac{1}{L_\varepsilon(N)}\right)\right\}, \tag{19}$$

where $\sigma = \Re(s)$, $\tau = \Im(s)$, and $\widehat{\rho}$ is the Laplace transform of the Dickman function.

Since $\lim_{s\to 1}(s-1)\zeta(s) = 1$, we see from Lemma 5.1 that $\zeta(s, N)$ behaves like $\log N$ "near" $s = 1$. We aim to apply Lemma 5.1 to the partial zeta function $\zeta\left(1 + \frac{ix}{\log N}, N\right)$ in the integral (10). Therefore $s = 1 + \frac{ix}{\log N}$ with $x \in \mathbb{R}$, and hence $\sigma = 1$ and $\tau = \frac{x}{\log N}$. The second condition in (18) is trivially satisfied, while the third condition reads as

$$|x| \leq (\log N)\exp\{(\log N)^{3/5-\varepsilon}\}. \tag{20}$$

It is also worthwhile mentioning that an improved version of Lemma 5.1 is available in [13] (Lemma 5.16 on page 531), relying on finer analysis of the zero-free region for the zeta function. However, since we apply it to the case of $\sigma = 1$, the improved version is not needed in our analysis.

We will also need the following classical estimate of the size of the Riemann zeta function along the $\sigma = 1$ line. Namely, the fact (due to Vinogradov and Korobov) that there is a constant $A > 0$ such that for every $|t| \geq 3$ we have

$$|\zeta(1+it)| \leq A(\log|t|)^{2/3}, \tag{21}$$

see, e.g., Lemma 8.28 in [9]. Ford proved in [4] that we can take $A = 76.2$ in (21). We can therefore get the bounds on the $\sigma = 1$ line:

$$1 \ll |\zeta(1+it)(it)| \ll \begin{cases} 1 & \text{if } |t| \leq 3; \\ |t|(\log|t|)^{2/3} & \text{if } |t| \geq 3. \end{cases} \tag{22}$$

6. Isolating the main term

We now split the integral in (10) so that we can apply Lemma 5.1. The main term in our sum (1) will come from considering $|x| \leq 3\log N$, which is allowed by (20) provided N is sufficiently large. We write $S_{\Omega,f}(z, k; N)$ as

$$\int_{-\infty}^{\infty} \widehat{f}(x)\,\zeta\left(1 + \frac{ix}{\log N}, N\right)^z h_{z,k,N}\left(1 + \frac{ix}{\log N}\right)dx = \int_{|x|\leq 3\log N} + \int_{|x|>3\log N} =: \mathcal{I}_1 + \mathcal{I}_2. \tag{23}$$

Using Lemma 5.1 as discussed in Section 5., we obtain

$$\mathcal{I}_1 = (\log N)^z \int_{|x|\leq 3\log N} \widehat{f}(x)\,\zeta\left(1 + \frac{ix}{\log N}\right)^z \left(\frac{ix}{\log N}\right)^z \widehat{\rho}(ix)^z\, h_{z,k,N}\left(1 + \frac{ix}{\log N}\right)dx + E_1 \tag{24}$$

where, by the discussion in Section 3.,

$$\begin{aligned} |E_1| &\ll \frac{(\log N)^{\Re(z)}}{L_\varepsilon(N)} \int_{|x|\leq 3\log N} \left|\widehat{f}(x)\,\zeta\left(1 + \frac{ix}{\log N}\right)^z \left(\frac{ix}{\log N}\right)^z \widehat{\rho}(ix)^z\, h_{z,k,N}\left(1 + \frac{ix}{\log N}\right)\right| dx \\ &\ll \frac{(\log N)^{\Re(z)}}{L_\varepsilon(N)} \int_{|x|\leq 3\log N} \left|\widehat{f}(x)\,\zeta\left(1 + \frac{ix}{\log N}\right)^z \left(\frac{ix}{\log N}\right)^z \widehat{\rho}(ix)^z\right| dx. \end{aligned} \tag{25}$$

We will estimate E_1 and $\mathcal{I}_2$ in the Section 7. and show that they are $o((\log N)^{\Re(z)})$ as $N \to \infty$. The dominant behaviour of the sum $S_{\Omega,f}(z,k;N)$ is therefore given by the first term in the right-hand-side of (24):

$$(\log N)^z \int_{|x|\le 3\log N} \widehat{f}(x)\,\zeta\left(1+\frac{ix}{\log N}\right)^z \left(\frac{ix}{\log N}\right)^z \widehat{\rho}(ix)^z\; h_{z,k,N}\left(1+\frac{ix}{\log N}\right)dx. \tag{26}$$

Note that this gives a main term since the integral is $O(1)$. In fact, by (5), (17), and (22), we obtain the estimate

$$\begin{aligned}\left|\int_{|x|\le 3\log N} \widehat{f}(x)\,\zeta\left(1+\frac{ix}{\log N}\right)^z \left(\frac{ix}{\log N}\right)^z \widehat{\rho}(ix)^z\; h_{z,k,N}\left(1+\frac{ix}{\log N}\right)dx\right|\\ \ll \int_{|x|\le 3\log N} \frac{dx}{(1+x^2)^{\frac{\eta+\Re(z)}{2}}} \ll 1,\end{aligned} \tag{27}$$

provided $\eta+\Re(z)>1$.

We can further simplify our main term (26) by replacing $h_{z,k,N}(1+\frac{ix}{\log N})$ by $h_{z,k}(1+\frac{ix}{\log N})$. In fact, by Lemma 3.1, (26) equals

$$(\log N)^z \int_{|x|\le 3\log N} \widehat{f}(x)\,\zeta\left(1+\frac{ix}{\log N}\right)^z \left(\frac{ix}{\log N}\right)^z \widehat{\rho}(ix)^z\; h_{z,k}\left(1+\frac{ix}{\log N}\right)dx + E_2, \tag{28}$$

where

$$|E_2| \ll \frac{(\log N)^{\Re(z)-1}}{N}. \tag{29}$$

One of the advantages of using Lemma 5.1 is its range of applicability (20), which we use when we consider the region $|x|\le 3\log N$ in (26). In comparison, the probabilistic approach in [2] only provides a main term where $(\log N)^z$ is multiplied by an integral over a region of the form $|x|\le R(N)$ with $R(N)=o(\log N)$ as $N\to\infty$; see also the Remark before Section 5.2 in [2].

7. Estimating the error terms

Let us estimate the first error term E_1. Using the same argument as in (27) for the integral in (25) we obtain, for every $\varepsilon>0$,

$$|E_1| \ll (\log N)^{\Re(z)} \exp\{-(\log N)^{3/5-\varepsilon}\} \tag{30}$$

for all sufficiently large N. Let us now estimate the integral $\mathcal{I}_2$ from (23).

If $\Re(z)<0$, then we can use the lower bound $|\zeta(1+i\tau,N)|\gg 1$ (uniform in $N\ge 1$), (5), and (17) to obtain

$$\begin{aligned}|\mathcal{I}_2| &\le \int_{|x|>3\log N}\left|\widehat{f}(x)\,\zeta\left(1+\frac{ix}{\log N},N\right)^z h_{z,N}\left(1+\frac{ix}{\log N}\right)\right|dx\\ &\ll \int_{|x|>3\log N}\frac{dx}{|x|^\eta} \ll (\log N)^{-\eta+1}.\end{aligned} \tag{31}$$

On the other hand, if $\Re(z)\ge 0$, we use (17), Lemma 5.1, our assumption (5), and the upper bound in (22). We get

$$\begin{aligned}|\mathcal{I}_2| &\le \int_{|x|>3\log N}\left|\widehat{f}(x)\,\zeta\left(1+\frac{ix}{\log N},N\right)^z h_{z,N}\left(1+\frac{ix}{\log N}\right)\right|dx\\ &\ll (\log N)^{\Re(z)}\int_{|x|>3\log N}\frac{1}{|x|^\eta}\left|\zeta\left(1+\frac{ix}{\log N}\right)^z\left(\frac{ix}{\log N}\right)^z\widehat{\rho}(ix)^z\right|dx\\ &\ll (\log(N))^{\Re(z)}\int_{|x|>3\log N}\frac{1}{|x|^{\eta+\Re(z)}}\left|\frac{x}{\log N}\right|^{\Re(z)}\left(\log\left|\frac{x}{\log N}\right|\right)^{\frac{2\Re(z)}{3}}dx\\ &\ll \int_{|x|>3\log N}\frac{(\log|x|)^{\frac{2\Re(z)}{3}}}{|x|^\eta}dx \ll (\log(N))^{-\eta+1}(\log\log N)^{\frac{2\Re(z)}{3}}.\end{aligned} \tag{32}$$

8. The main theorem

Combining (23), (24), (27), (28), (29), (30), (31), and (32), we obtain the following

Theorem 8.1. *Fix $z \in \mathbb{C}$ and an integer $k \geq 2$. Suppose that f satisfies* (5) *with*

$$\eta > \max\{1, 1 - \Re(z)\}. \tag{33}$$

Then

$$S_{\Omega,f}(z, k; N) = C_f(z, k; N)\, \log(N)^z (1 + E(N)), \tag{34}$$

where

$$C_f(z, k; N) := \int_{|x| \leq 3 \log N} \widehat{f}(x)\, \widehat{\rho}(ix)^z\, \zeta\left(1 + \frac{ix}{\log N}\right)^z \left(\frac{ix}{\log N}\right)^z h_{z,k}\left(1 + \frac{ix}{\log N}\right) dx, \tag{35}$$

$\widehat{\rho}$ is the Laplace transform of the Dickman function, and $h_{z,k}$ is defined in (11). *Moreover, as $N \to \infty$, we have that $C_f(z, k; N) = O(1)$ and*

$$E(N) = \begin{cases} O\big((\log N)^{-\eta+1}\big) & \text{if } \Re(z) < 0, \\ O\big((\log N)^{-\eta+1} (\log \log N)^{\frac{2\Re(z)}{3}}\big) & \text{if } \Re(z) \geq 0. \end{cases} \tag{36}$$

Theorem 1.1 follows immediately from Theorem 8.1 if we assume that f is of Schwartz class, since η can be taken arbitrarily large and hence the error term (36) is $O_\theta(\log^{-\theta} N)$ for every $\theta > 0$.

9. Remarks on $C_f(z, k; N)$

Recalling (16), we observe that the function

$$x \mapsto F_{f,z}(x) := \widehat{f}(x) \widehat{\rho}(ix)^z = \frac{\widehat{f}(x)}{(ix)^z} \exp\left(-z \int_0^\infty \frac{e^{-ix-t}}{ix+t} dt\right)$$

can be interpreted as the Fourier transform of the convolution of f with the z-convolution of the Dickman function ρ. Although this is a priori only a distribution in the sense of Schwartz, the assumption (33) ensures that it is actually a function and that it decays to 0 in absolute value as $|x| \to \infty$. The change of variables $\tau = \frac{x}{\log N}$ yields

$$C_f(z, k, N) = \log N \int_{-3}^{3} F_{f,z}(\tau \log N) \zeta(1 + i\tau)^z (i\tau)^z h_{z,k}(1 + i\tau)\, d\tau. \tag{37}$$

We already know that the quantity $C_f(z, k, N)$ is $O(1)$ as $N \to \infty$. In order to find its asymptotic value as $N \to \infty$ ignoring error terms, we may consider the contribution to the integral in (37) of a neighbourhood of $\tau = 0$ of size $O(\frac{1}{\log N})$. In this neighbourhood, the part of the integrand given by $\zeta(1 + i\tau)^z (i\tau)^z h_{z,k}(1 + i\tau)$ can be approximated by its limit as $\tau \to 0$, i.e.

$$c(z, k) := \prod_p \left(1 + \frac{z}{p} + \frac{z^2}{p^2} + \cdots + \frac{z^{k-1}}{p^{k-1}}\right) \left(1 - \frac{1}{p}\right)^z, \tag{38}$$

provided this product does not vanish. Observe that the Euler product (38) vanishes when $z = pe^{2\pi i \frac{m}{k}}$ for some prime p and some $m \in \{1, 2, \ldots, k-1\}$. Therefore, given $k \geq 2$, if z is not the product of a prime and a nontrivial k-th root of unity, then we have

$$C_f(z, k; N) \sim c(z, k) \int_{-3 \log N}^{3 \log N} F_{f,z}(t)\, dt. \tag{39}$$

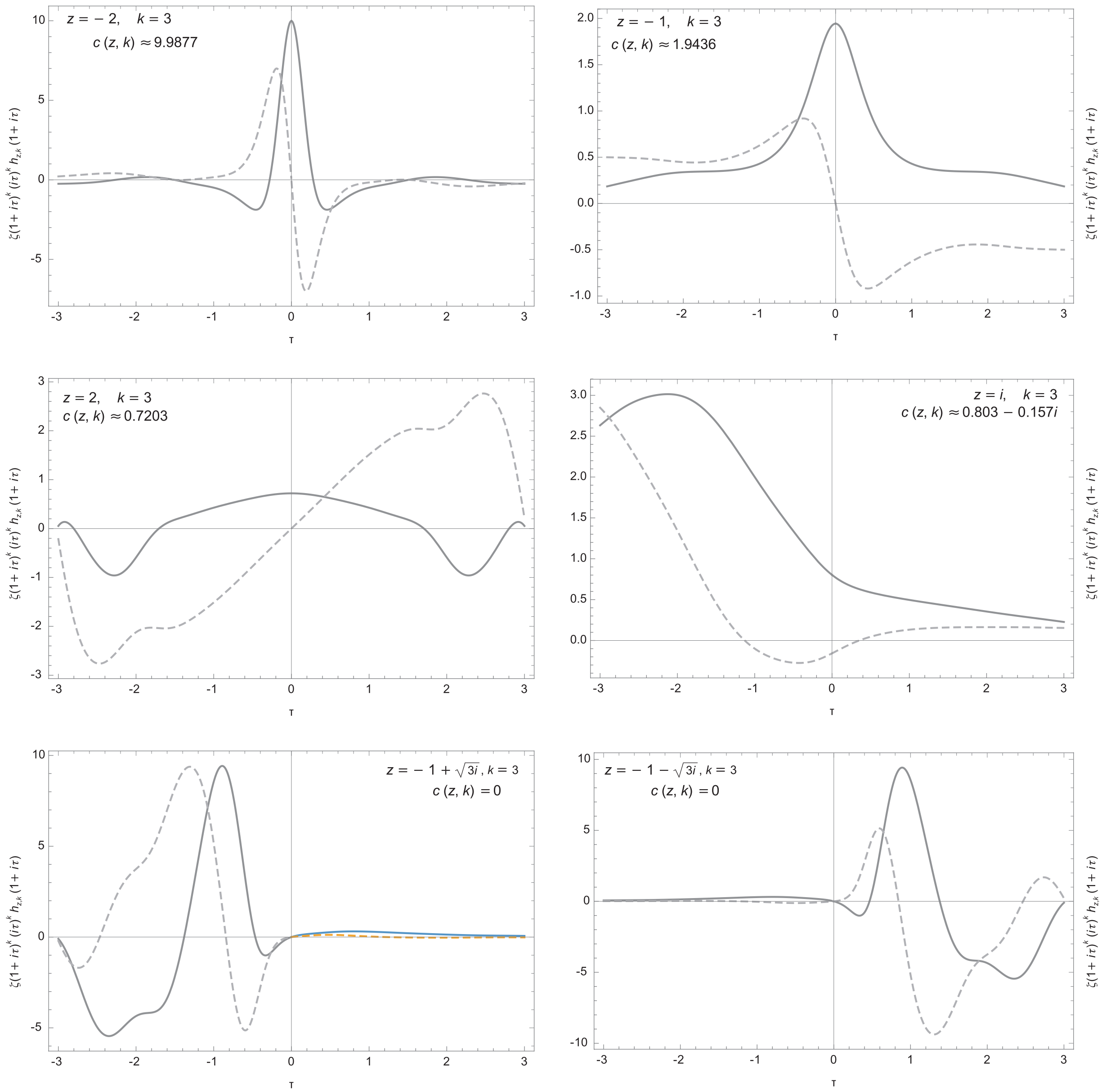

Figure 1. The function $\tau \mapsto \zeta(1+i\tau)^z (i\tau)^z h_{z,k}(1+i\tau)$ for $k = 3$ and six different values of z (indicated in each panel). The solid line represents the real part of the function, while the dashed line represents its imaginary part. In each panel, the constant (38) is shown. In the bottom panels $c(-1 \pm \sqrt{3}, 3) = 0$ because $z = -1 \pm \sqrt{3} = 2e^{2\pi i \frac{\pm 1}{3}}$.

In applications, it is therefore important to understand the function $F_{f,z}$ in order to find the asymptotic value of $C_f(z, k, N)$ as $N \to \infty$.

Consider for instance the case $k = 3$. The function $\tau \mapsto \zeta(1+i\tau)^z (i\tau)^z h_{z,3}(1+i\tau)$ is shown in Figure 1 for various values of z.

If we look at the case $z = -1$ (top right panel in Figure 1) and $f(t) = e^{-\frac{t^2}{2}}$, we have $c(-1, 3) = \prod_p \left(1 + \frac{1}{p(p-1)}\right) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \approx 1.943596437$ and where $\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt$ and $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt$. In this case the imaginary part of the integrand $F_{f,-1}(x)$ is odd and does not contribute, while $\Re(F_{f,-1}(x)) = \frac{x}{\sqrt{2\pi}} e^{-\mathrm{Ci}(x) - \frac{x^2}{2}} \cos(\mathrm{Si}(x))$ is even. Therefore, as $N \to \infty$,

$$C_f(-1, 3; N) \sim \frac{2\,\zeta(2)\zeta(3)}{\zeta(6)} \int_0^{3 \log N} \frac{x}{\sqrt{2\pi}} e^{-\mathrm{Ci}(x) - \frac{x^2}{2}} \cos(\mathrm{Si}(x)) dx \tag{40}$$

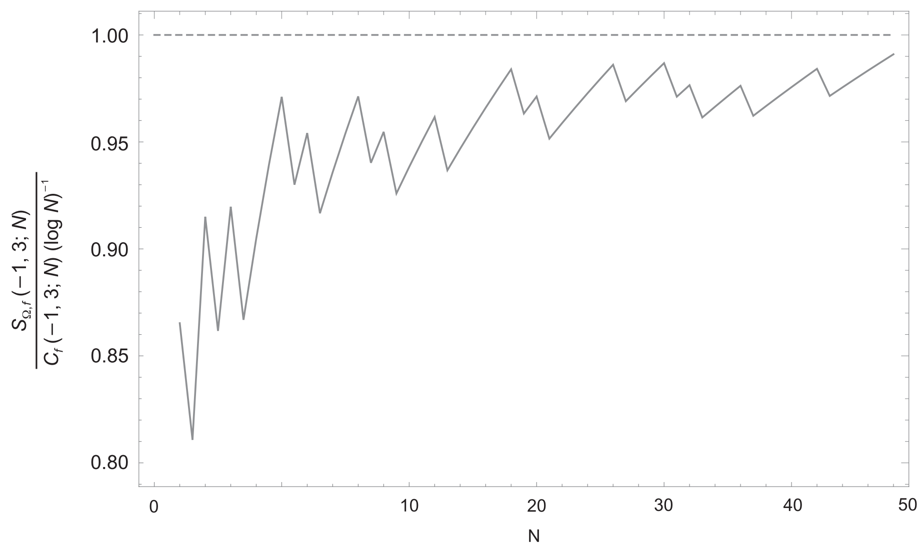

Figure 2. The ratio $\frac{S_{\Omega,f}(-1,3;N)}{C_f(-1,3;N)(\log N)^{-1}}$ for $f(t) = e^{-\frac{t^2}{2}}$ and $2 \le N \le 58$. Note that the sum $S_{\Omega,f}(-1,3;58)$ consists of $3^{\pi(58)} \approx 4.3 \times 10^7$ terms.

and the integral in (40) can be seen to converge very rapidly as $N \to \infty$ to a constant – which can be estimated to any precision – whose approximate value is 0.2071881. Recalling that the Liouville function can be written as $\lambda(n) = (-1)^{\Omega(n)}$, our result therefore yields that

$$S_{\Omega,f}(-1,3;N) = \sum_{\substack{n \text{ is } 3\text{-free} \\ p|n \Rightarrow p \le N}} \exp\left(-\frac{1}{2}\left(\frac{\log n}{\log N}\right)^2\right)\frac{\lambda(n)}{n} \sim 0.80538\cdots(\log N)^{-1}$$

as $N \to \infty$.

We can further illustrate our result for small values of N, when the set $\{n \in \mathbb{N} : k\text{-free}, \ p|n \Rightarrow p \le N\}$ and the sum $S_{\Omega,f}(z,k;N)$ can be computed explicitly, while (37) can be evaluated numerically, provided an expression for $\widehat{f}$ – and hence $F_{f,z}(x)$ – is available. In concrete cases, we see that the ratio $\frac{S_{\Omega,f}(z,k;N)}{C_f(z,k;N)(\log N)^z}$ is quite close to 1, as predicted by Theorem 1.1 for large N. See Figure 2 for the case $z = -1$, $k = 3$ discussed above.

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