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Volume 131, Issue 1, January 2011 ISSN 0022-314X
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Journal of Number Theory 131 (2011) 1723–1733

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A problem of Chowla revisited ☆

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A R T I C L E   I N F O

Article history:
Received 22 July 2010
Revised 31 March 2011
Accepted 3 April 2011
Available online xxxx
Communicated by Michael A. Bennett

MSC:
11J81
11J86

Keywords:
Chowla's problem
Application of Baker's method
Special values of Dirichlet L-series

A B S T R A C T

In 1964, S. Chowla asked if there is a non-zero integer-valued function f with prime period p such that $f(p) = 0$ and

$$\sum_{n=1}^{p} f(n) = 0$$

and $\sum_{n=1}^{\infty} f(n)/n = 0$? Chowla conjectured that there was no such function. Later that year, Chowla proved a special case of his conjecture and in a subsequent paper, asked if the condition that $f(p) = 0$ can be dropped. In 1973, Baker, Birch and Wirsing, using the theory of linear forms in logarithms obtained a general theorem, a special case of which implies the conjecture of Chowla. They alluded to the fact that Chowla also settled the conjecture in his special case, but this proof (if different from the one by Baker, Birch and Wirsing) never seems to have been published. We resurrect the 1964 approach of Chowla and indicate how the cyclotomic units discovered by Ramachandra in 1966, can be combined with Baker's theorem to answer Chowla's question. We also obtain a mild generalization of the theorem of Baker, Birch and Wirsing.

☆ Research of both authors partially supported by NSERC Discovery grants.

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0022-314X/$ – see front matter © 2011 Published by Elsevier Inc.
doi:10.1016/j.jnt.2011.04.003

1. Introduction

In a paper written in 1964, Sarvadaman Chowla [3] considered the following problem. Let p be a prime number and f an integer-valued arithmetical function, not identically zero, with period p, such that $f(p) = 0$ and

$$\sum_{n=1}^{p} f(n) = 0.$$

Chowla conjectured that under these conditions,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

Following an argument outlined by Siegel, Chowla [3] proved this conjecture in the case that f is odd (that is, $f(-n) = -f(n)$). Since his argument is very short and elegant, we give it below (see Section 2). At the same time, we note that the argument applies in a wider context with minor changes.

In a later paper, Chowla [4] asked if there exists a rational valued function $f : \mathbb{Z} \to \mathbb{Q}$ with prime period q such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0. \tag{1}$$

The difference now is that $f(q)$ is not required to be zero. One can also investigate the general case when q is not necessarily prime and inquire under what conditions the sum (1) is zero. This general question was addressed by Baker, Birch and Wirsing [2] using Baker's theory of linear forms in logarithms. If we require that the function also satisfies the condition

$$f(a) = 0 \quad \text{for } 1 < (a, q) < q, \tag{2}$$

then they showed that there is no such function for arbitrary q.

More generally, if f takes values in an algebraic number field K which is disjoint from the qth cyclotomic field, and satisfies (2), then they proved that no such function exists.

After describing Chowla's original question, Baker, Birch and Wirsing wrote in a footnote on page 225 of their paper [2] that "While working on the manuscript, we were informed by Professor Chowla that he had also solved the problem to the extent stated above." It is unclear what the phrase "to the extent stated above" means but judging from the context, it seems to mean that Chowla also solved the problem in the case q is prime. However, Chowla does not seem to have published any work on this to give us an indication of what his methods were, if they were different from those of [2].

In [5], we indicated how Chowla may have proved his theorem using tools at his disposal at that time. Using ideas of this paper, we show that a new simplified proof of the Baker–Birch–Wirsing theorem can be given. We also give a modest extension of the theorem of Baker, Birch and Wirsing [2]. It will be convenient to define the Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

As indicated in [7], this series admits an analytic continuation to the entire complex plane by virtue of the theory of the Hurwitz zeta function.

Theorem 1. *Let $f : \mathbb{Z}/q\mathbb{Z} \to \overline{\mathbb{Q}}$ be such that*

$$\sum_{a=1}^{q} f(a) = 0,$$

and $f(a) = 0$ for $1 < (a, q) < q$. Write $f = f_e + f_o$ where f_e is an even function and f_o is an odd function. If the qth cyclotomic polynomial is irreducible over the field generated by the values of f_o, then $L(1, f) \neq 0$ unless f is identically zero.

As a corollary, we deduce

Corollary 2. *The values $L(1, \chi)$ as χ ranges over the even Dirichlet characters (mod q) are linearly independent over $\overline{\mathbb{Q}}$.*

The corollary is not true for odd characters since in that case, each $L(1, \chi)$ is equal to an algebraic multiple of π. Moreover, we remark that Corollary 2 together with Schanuel's conjecture implies the algebraic independence of the $L(1, \chi)$ as χ ranges over the even Dirichlet characters (mod q).

2. Group-theoretic preliminaries

We begin with a straightforward result from group theory which is an interesting variant of Artin's theorem on the linear independence of the irreducible characters of a finite group G. As usual, we can define an inner product on the space $C(G)$ of complex-valued functions on G. Indeed, if $f.g \in C(G)$, then

$$(f, g) = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}.$$

Lemma 3. *Let G be a finite group. Suppose that*

$$\sum_{\chi \neq 1} \chi(R) u_\chi = 0$$

for all $R \neq 1$ and all irreducible characters $\chi \neq 1$ of G. Then $u_\chi = 0$ for all $\chi \neq 1$.

Proof. For any irreducible character $\psi \neq 1$, we can multiply our equation by $\overline{\psi(R)}/|G|$ and sum over $R \neq 1$ to obtain

$$0 = \frac{1}{|G|} \sum_{R \neq 1} \overline{\psi(R)} \sum_{\chi \neq 1} \chi(R) u_\chi = \sum_{\chi \neq 1} u_\chi \left((\psi, \chi) - \frac{\psi(1)\chi(1)}{|G|} \right).$$

Thus, by the orthogonality relations,

$$0 = u_\psi - \frac{\psi(1)}{|G|} \sum_{\chi \neq 1} u_\chi \chi(1) = u_\psi - \frac{\psi(1)}{|G|} S \quad \text{(say)}.$$

Hence, for every $R \neq 1$, we have

$$0 = \frac{1}{|G|} \sum_{\chi \neq 1} \chi(R)\chi(1)S.$$

Recalling that

$$\frac{1}{|G|} \sum_{\chi} \chi(R)\chi(1) = 0$$

unless $R = 1$, we deduce that $S = 0$. Hence $u_\chi = 0$ for all $\chi \neq 1$ as desired. □

3. Baker's theorem and variations

In the discussion below, a pivotal role is played by the fundamental theorem of Baker concerning linear forms in logarithms. We record this as:

Lemma 4. *If* $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}} \backslash \{0\}$ *and* $\beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}}$*, then*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental. The latter case arises if $\log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over* $\mathbb{Q}$ *and* $\beta_1, \ldots, \beta_n$ *are not all zero.*

Proof. This is the content of Theorems 2.1 and 2.2 of [1]. Let us note that here and later, we interpret log as the principal value of the logarithm with the argument lying in the interval $(-\pi, \pi]$. □

In particular, if $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$, then they are linearly independent over $\overline{\mathbb{Q}}$.

As an application of Lemma 4, we prove the following variant of a result from [9].

Lemma 5. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *be positive algebraic numbers. If* $c_0, c_1, \ldots, c_n$ *are algebraic numbers with* $c_0 \neq 0$*, then*

$$c_0\pi + \sum_{j=1}^{n} c_j \log \alpha_j$$

is a transcendental number and hence non-zero.

Proof. Let S be such that $\{\log \alpha_j\colon\ j \in S\}$ be a maximal $\mathbb{Q}$-linearly independent subset of

$$\log \alpha_1, \ldots, \log \alpha_n.$$

We write $\pi = -i \log(-1)$. We can re-write our linear form as

$$-ic_0 \log(-1) + \sum_{j \in S} d_j \log \alpha_j,$$

for algebraic numbers d_j. By Baker's theorem, this is either zero or transcendental. The former case cannot arise if we show that

$$\log(-1), \quad \log\alpha_j, \quad j \in S$$

are linearly independent over $\mathbb{Q}$. But this is indeed the case since

$$b_0 \log(-1) + \sum_{j\in S} b_j \log\alpha_j = 0$$

for integers b_0, b_j, $j \in S$ implies that

$$\prod_{j\in S} \alpha_j^{2b_j} = 1,$$

which in turn implies $b_j = 0$ for all $j \in S$ since α_j, for $j \in S$ are multiplicatively independent. Consequently, $b_0 = 0$. This completes the proof. □

4. Revisiting the problem of Chowla

We now give a different approach to the problem of Chowla [3,4] and obtain a generalization of the theorem of Baker, Birch and Wirsing [2]. Consider an algebraic-valued function f defined on the residue classes (mod q). We assume that $f(0) = 0$ and that

$$\sum_{a=1}^{q} f(a) = 0.$$

Our strategy is to write

$$f = f_e + f_o$$

where f_e is even (that is, $f(-n) = f(n)$) and f_o is odd (that is, $f(-n) = -f(n)$). We can write

$$f = \sum_{\chi\neq 1} c_\chi \chi,$$

where the sum is over non-trivial Dirichlet characters χ (mod q). The trivial character does not appear since

$$\sum_{a=1}^{q} f(a) = 0.$$

Thus, we have

$$f_e = \sum_{\chi \text{ even}, \, \chi\neq 1} c_\chi \chi,$$

and

$$f_o = \sum_{\chi \text{ odd}} c_\chi \chi.$$

We recall that Ramachandra (see Theorem 8.3 on page 147 of [12] as well as [11]) discovered a set of real multiplicatively independent units in the cyclotomic field, which we will denote by ξ_a (with $1 < a < q/2$ and $(a, q) = 1$) using the notation of [12]. A fundamental property of these units is the following formula (see the proof of Theorem 8.3 on page 149 in [12]): for even χ with $\chi \neq 1$, we have

$$L(1, \chi) = A_\chi \sum_{1<a<q/2} \overline{\chi}(a) \log \xi_a, \tag{3}$$

where A_χ is a non-zero algebraic number. This is the cyclotomic analogue of one of the main theorems of [10] in the case of an imaginary quadratic field (see also [11]) we need in our context. To elaborate, let ζ be a primitive qth root of unity and following Ramachandra [10], define

$$\eta_a = \prod_{d|q,\, d\neq q,\, (d,q/d)=1} \frac{1-\zeta^{ad}}{1-\zeta^d}.$$

Setting $d_a = \frac{1}{2}(1-a)\sum_{d|q,(d,q/d)=1,\, d\neq q} d$, one sees that $\xi_a = \zeta^{d_a}\eta_a$ lies in the real subfield $\mathbb{Q}(\zeta + \zeta^{-1})$. These are the multiplicatively independent units for $1 < a < q/2$ with $(a, q) = 1$. Following the calculation on page 149 in [12], we see that

$$\sum_{a=1}^{q} \overline{\chi}(a) \sum_{d|q,\, (d,q/d)=1,\, d\neq q} \log\left|1-\zeta_q^{ad}\right|$$

is a non-zero algebraic multiple of $L(1, \chi)$. This easily leads to the formula (3) above. Proceeding as in the case of the imaginary quadratic field [6], we have that

$$\begin{aligned} L(1, f_e) &= \sum_{\chi \text{ even},\, \chi\neq 1} c_\chi L(1, \chi) \\ &= \sum_{\chi \text{ even},\, \chi\neq 1} c_\chi A_\chi \left(\sum_{1<a<q/2} \overline{\chi}(a) \log \xi_a \right) = \sum_{1<a<q/2} \left(\sum_{\chi \text{ even},\, \chi\neq 1} A_\chi c_\chi \overline{\chi}(a) \right) \log \xi_a. \end{aligned}$$

Since the ξ_a's are multiplicatively independent, the $\log \xi_a$'s are linearly independent over $\mathbb{Q}$. By Baker's theorem, they are linearly independent over $\overline{\mathbb{Q}}$. Consequently, $L(1, f_e) = 0$ if and only if

$$\sum_{\chi \text{ even},\, \chi\neq 1} A_\chi c_\chi \overline{\chi}(a) = 0, \quad 1 < a < q/2.$$

Now the even characters of $(\mathbb{Z}/q\mathbb{Z})^*$ can be viewed as characters of the group $(\mathbb{Z}/q\mathbb{Z})^*/\{\pm 1\}$ and by Lemma 3, we deduce that $c_\chi = 0$ for all even χ. In other words, we have proved the following theorem:

Theorem 6. *Let f_e be an even algebraic valued function and suppose that $\sum_{a=1}^{q} f_e(a) = 0$ and $f_e(0) = 0$. Then $L(1, f_e) \neq 0$ unless f_e is identically zero. Moreover, $L(1, f_e)$ is an algebraic linear combination of logarithms of multiplicatively independent units of the qth cyclotomic field. In particular, if f_e is not identically zero, then $L(1, f_e)$ is transcendental.*

As an immediate corollary, we deduce (a seemingly new result in the classical case):

Corollary 7. *$L(1,\chi)$, as χ ranges over non-trivial even characters mod q, are linearly independent over $\overline{\mathbb{Q}}$.*

The Chowla–Siegel theorem, to be outlined in the next section, shows that $L(1, f_o)$ is an algebraic multiple of π. Moreover, if we assume that the field generated by the values of f_o is disjoint from the qth cyclotomic field, then it shows that $L(1, f_o) \neq 0$ unless f_o is identically zero. Consequently, combining this result with Theorem 6 and then invoking Lemma 5, we deduce that $L(1, f) = 0$ implies that $L(1, f_o) = 0$ which can happen only if f_o is identically zero. In the latter case, $f = f_e$ is even and we have $L(1, f_e) \neq 0$ unless f_e is identically zero. This proves:

Theorem 8. *Let $f : (\mathbb{Z}/q\mathbb{Z})^* \to \overline{\mathbb{Q}}$ and suppose that $f(0) = 0$ and $\sum_{a=1}^{q} f(a) = 0$. Let K be the field generated by the values of f_o. If K is disjoint from the qth cyclotomic field, then $L(1, f) \neq 0$ unless f is identically zero.*

To complete our discussion, we give a proof of the theorem of Chowla and Siegel in the next section.

5. The Chowla–Siegel theorem

We begin by giving the proof (with some minor variations) of Chowla and Siegel [3] when f is an odd function. The essential idea here is to use the familiar cotangent expansion:

$$\pi \cot \pi x = \sum_{n\in\mathbb{Z}} \frac{1}{n+x}.$$

Theorem 9. *Let K be an algebraic number field which is disjoint from the qth cyclotomic field. Let $f : \mathbb{Z}/q\mathbb{Z} \to K$ be odd, that is, $f(-n) = -f(n)$. Suppose that $f(n) = 0$ whenever $(n,q) > 1$. Then,*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0,$$

unless f is identically zero.

Remark. As noted earlier, only the case q prime and K is the rational number field is considered in [3]. However, a careful study shows that their argument gives a proof of the theorem stated above.

Proof of Theorem 9. Let

$$S_k = \sum_{n\in\mathbb{Z},\, n\neq 0} \frac{f(kn)}{n}.$$

Since f is odd,

$$S_1 = 2\sum_{n=1}^{\infty} \frac{f(n)}{n}.$$

We will show that if $S_1 = 0$, then f is identically zero. To this end, we observe that

$$S_k = \sum_{a \,(\mathrm{mod}\, q)} f(ka) \sum_{n\equiv a \,(\mathrm{mod}\, q),\, n\neq 0} \frac{1}{n}$$

and the inner sum is

$$\frac{1}{q}\sum_{t\in\mathbb{Z}}\frac{1}{t+a/q}=\frac{\pi}{q}\cot\frac{\pi a}{q}=\frac{2\pi i}{q}\left(\frac{1}{2}+\frac{1}{\zeta^a-1}\right),$$

where $\zeta = e^{2\pi i/q}$. Thus,

$$-iS_k=\sum_{a\,(\mathrm{mod}\,q)} f(ka)\frac{2\pi}{q}\left(\frac{1}{2}+\frac{1}{\zeta^a-1}\right).$$

Since k is coprime to q we have

$$\sum_{(a,q)=1} f(ka)=\sum_{(a,q)=1} f(a)=0,$$

as the sum can be taken over all $a \pmod q$ and

$$\sum_{(a,q)=1} f(a)=\sum_{(a,q)=1} f(-a)=-\sum_{(a,q)=1} f(a).$$

Thus, the first sum in the above expression for S_k disappears and we deduce that

$$-\frac{iqS_k}{2\pi}=\sum_{a\,(\mathrm{mod}\,q)}\frac{f(ka)}{\zeta^a-1}.$$

In particular,

$$-\frac{iqS_1}{2\pi}=\sum_{a\,(\mathrm{mod}\,q)}\frac{f(a)}{\zeta^a-1}.$$

This calculation in particular evaluates $L(1,\chi)$ when χ is odd. Now the right hand side is an algebraic number. For $(k,q)=1$, applying the Galois automorphism $\zeta\mapsto\zeta^{k'}$ where $kk'\equiv 1 \pmod q$, we see that $S_1=0$ implies $S_k=0$ for all $(k,q)=1$. Note that this step is valid if f is K-valued and K is disjoint from the qth cyclotomic field. Hence, if $S_1=0$, then $S_k=0$ for all $(k,q)=1$ and so

$$0=\sum_{k\,(\mathrm{mod}\,q)}\overline{\chi}(k)S_k=\sum_{k\,(\mathrm{mod}\,q)}\overline{\chi}(k)\sum_{a\,(\mathrm{mod}\,q)}\frac{f(ka)}{\zeta^a-1}=\sum_{a,k\,(\mathrm{mod}\,q)} f(ka)\frac{\overline{\chi}(ka)\chi(a)}{\zeta^a-1}.$$

Now put $ka\equiv b \pmod q$ to obtain

$$0=\sum_{b\,(\mathrm{mod}\,q)} f(b)\overline{\chi(b)}\sum_{k\,(\mathrm{mod}\,q)}\frac{\chi(k'b)}{\zeta^{k'b}-1}.$$

For fixed b, the number $k'b$ runs over all coprime residue classes $(\mathrm{mod}\ q)$ as k runs over all coprime residue classes $(\mathrm{mod}\ q)$. Thus, the inner sum is

$$\sum_{t\,(\mathrm{mod}\,q)}\frac{\chi(t)}{\zeta^t-1}=\frac{iqL(1,\chi)}{\pi}$$

for χ odd. In particular, by virtue of the fact that $L(1,\chi) \neq 0$, we deduce that for χ odd,

$$\sum_{b \ (\mathrm{mod}\ q)} f(b)\overline{\chi}(b) = 0. \tag{4}$$

Now (4) is also true if χ is even since

$$\sum_{b \ (\mathrm{mod}\ q)} f(b)\overline{\chi}(b) = \sum_{b \ (\mathrm{mod}\ q)} f(-b)\overline{\chi}(-b) = -\sum_{b \ (\mathrm{mod}\ q)} f(b)\overline{\chi}(b).$$

In any case, we have

$$\sum_{b \ (\mathrm{mod}\ q)} f(b)\chi(b) = 0$$

for all $\chi \pmod q$. Thus,

$$0 = \sum_{\chi} \chi(a)\Biggl(\sum_{b \ (\mathrm{mod}\ q)} f(b)\overline{\chi}(b)\Biggr) = \sum_{b \ (\mathrm{mod}\ q)} f(b) \sum_{\chi} \chi(a)\overline{\chi}(b) = \phi(q) f(a),$$

by the orthogonality relations. Hence f is identically zero. $\square$

As alluded to above, there are several noteworthy features in the above proof. First is that the argument works if f is K-valued and $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. The second is that the function is only supported on the coprime arguments. That is, f is “Dirichlet type” in the sense of [8].

6. Proof of Theorem 1

In the previous discussion, we assumed that f is supported only on the coprime residue classes mod q. In the theorem of Baker, Birch and Wirsing, $f(q) \neq 0$ is also admitted. We now show how a minor variation leads to Theorem 1.

Let g be a function defined on the residue classes mod q in the following way. $g(a) = 1$ if $(a,q) = 1$. We set $g(0) = -\phi(q)$ and $g(a) = 0$ for the remaining residue classes. In [7], the Hurwitz zeta function was used to explicitly evaluate $L(1, f)$ for $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$. If ψ denotes the digamma function (that is, the logarithmic derivative of the Γ-function), then

$$L(1, f) = -\frac{1}{q}\sum_{a=1}^{q} f(a)\psi(a/q).$$

Now $\psi(1) = -\gamma$, where γ is Euler's constant, and on page 311 of [7], we derived the formula

$$\sum_{a=1}^{q} \psi(a/q) = -q\log q - \gamma q. \tag{5}$$

To evaluate

$$\sum_{(a,q)=1} \psi(a/q),$$

we use the Möbius function:

$$\sum_{(a,q)=1} \psi(a/q) = \sum_{a=1}^{q} \psi(a/q) \sum_{d|q,a} \mu(d) = \sum_{d|q} \mu(d) \sum_{a_1=1}^{q/d} \psi\big(a_1/(q/d)\big).$$

Using (5), we see that this is equal to

$$\sum_{d|q} \mu(d)\left(-\frac{q}{d}\log\frac{q}{d} - \gamma\frac{q}{d}\right) = \sum_{d|q} \mu(q/d)(-d\log d - \gamma d).$$

Simplifying, and using our formula for $L(1, f)$ with f replaced by g,

$$L(1, g) = -\frac{1}{q}\sum_{a=1}^{q} g(a)\psi(a/q).$$

Since $g(0) = g(q) = -\phi(q)$, we have

$$qL(1, g) = -\phi(q)\gamma - \sum_{(a,q)=1} \psi(a/q).$$

The second sum on the right hand side is

$$\sum_{(a,q)=1} \psi(a/q) = \sum_{d|q} \mu(q/d)(-d\log d - \gamma d).$$

This is equal to

$$\sum_{(a,q)=1} \psi(a/q) = -\phi(q)\gamma - \sum_{d|q} \mu(q/d)\, d\log d.$$

Hence,

$$qL(1, g) = \sum_{d|q} \mu(q/d)\, d\log d.$$

Thus, we have

$$L(1, g) = \frac{1}{q}\sum_{d|q} \mu(q/d)\, d\log d.$$

We can rewrite this as

$$\frac{1}{q}\sum_{d|q} \mu(d)\left(\frac{q}{d}\log\frac{q}{d}\right) = \frac{\phi(q)}{q}\log q - \sum_{d|q} \frac{\mu(d)\log d}{d}.$$

It is not difficult to see that this is non-zero. This can be done in several ways. The easiest way is to observe that directly from the conditionally convergent series for $L(1, g)$, we have

$$L(1,g)=\sum_{j=1}^{\infty}\Biggl(\sum_{\substack{(j-1)q<a<jq\\(a,q)=1}}\frac{1}{a}-\frac{\phi(q)}{jq}\Biggr)=\sum_{j=1}^{\infty}S_j\quad\text{say.}$$

It is now clear that $S_j > 0$ for every j and $S_j \ll 1/j^2$. Now set

$$F(a)=-g(0)f(a)+f(0)g(a),$$

so that F is of Dirichlet type. Then,

$$L(1,F)=-g(0)L(1,f)+f(0)L(1,g).$$

If $L(1,f)=0$, then, $L(1,F)=f(0)L(1,g)$. On the other hand, $L(1,F)=L(1,F_e)+L(1,F_o)$ where $L(1,F_e)$ is a linear form in logarithms of units and $L(1,F_o)$ is an algebraic multiple of π. Since $L(1,g)$ is a linear form in logarithms of natural numbers, we have a contradiction unless $f(0)=0$. This reduces to the earlier case.

Acknowledgment

We thank the referee for helpful remarks on an earlier version of this paper.

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