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Finite Ramanujan expansions and shifted convolution sums of arithmetical functions, II

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ABSTRACT

We continue our study of convolution sums of two arithmetical functions f and g, of the form $\sum_{n\le N} f(n)g(n+h)$, in the context of heuristic asymptotic formulæ. Here, the integer $h \geq 0$ is called, as usual, the *shift* of the convolution sum. We deepen the study of finite Ramanujan expansions of general f, g for the purpose of studying their convolution sum. Also, we introduce another kind of Ramanujan expansion for the convolution sum of f and g, namely in terms of its shift h and we compare this "shift Ramanujan expansion", with our previous finite expansions in terms of the f and g arguments. Last but not least, we give examples of such shift expansions, in classical literature, for the heuristic formulæ.

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1. Introduction and statement of main results

We start, as in our previous paper, from the definition of the *Ramanujan sum* (see [22] and compare [20] for the properties):

$$c_q(n) \stackrel{def}{=} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \cos\left(\frac{2\pi a n}{q}\right) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e^{2\pi i a n/q} = \sum_{\substack{d|q \\ d|n}} d\mu\left(\frac{q}{d}\right) \tag{1}$$

(compare first three eqs. in [20]), where we abbreviate with (a,q) the greatest common divisor of any integers a and q, as usual, with μ the Möbius function: on primes $\mu(p) \stackrel{def}{=} -1$,

$$\mu(1) \stackrel{def}{=} 1, \qquad \mu(p_1 \cdots p_r) \stackrel{def}{=} (-1)^r$$

for $r \geq 2$ distinct primes p_j and $\mu(n) \stackrel{def}{=} 0$ on all other integers $n > 1$.

Given $f, g : \mathbb{N} \to \mathbb{C}$ any arithmetic functions, we may consider the *shifted convolution sum* of f and g, which we abbreviate as the *correlation* of f and g in the sequel, that we studied in our previous papers in this series

$$C_{f,g}(N,h) \stackrel{def}{=} \sum_{n \leq N} f(n) g(n+h),$$

where the integer $h \geq 0$ is called the *shift*. Under suitable conditions, we proved in [8] that

$$C_{f,g}(N,h) = \mathfrak{S}_{f,g}(h) N + O(N^{1-\delta}(\log N)^{4-2\delta}) \tag{2}$$

(compare Theorem 2 in [8] for the precise statement), for a $\delta > 0$, defining the *singular series* of f and g as

$$\mathfrak{S}_{f,g}(h) \stackrel{def}{=} \sum_{q=1}^{\infty} \widehat{f}(q) \widehat{g}(q) c_q(h),$$

where $\widehat{f}(q)$ and $\widehat{g}(q)$ are related to the *Ramanujan coefficients* of f and g respectively, see section 3 for the definition, then subsequent sections for the properties and examples.

However, we briefly give the *Ramanujan expansion* of any f, of coefficients $\widehat{f}(q)$,

$$f(n) \stackrel{def}{=} \sum_{q=1}^{\infty} \widehat{f}(q) c_q(n), \tag{3}$$

only assuming the pointwise convergence (compare Definition 2 in [7]).

We recall the vital remark we made in [8], in order to get, for fairly general f and g, *finite* Ramanujan expansions, namely, series like (3) to become finite sums. Defining for $f : \mathbb{N} \to \mathbb{C}$ the *Eratosthenes transform* (Aurel Wintner [24] coined this terminology), namely $f' \stackrel{def}{=} f * \mu$, so that Möbius inversion [2] gives: $f(n) = \sum_{d|n} f'(d)$ (likewise for g) and we have

$$C_{f,g}(N,h) = \sum_{d} f'(d) \sum_{q} g'(q) \sum_{\substack{n \le N \\ n \equiv 0 \bmod d \\ n \equiv -h \bmod q}} 1 = \sum_{d \le N} f'(d) \sum_{q \le N+h} g'(q) \sum_{\substack{n \le N \\ n \equiv 0 \bmod d \\ n \equiv -h \bmod q}} 1. \quad (4)$$

(It is suggestive to think of f' as the "arithmetical derivative" of f.) The above expression amounts to writing our arbitrary f, g as *truncated divisor sums*: see next section's (8), which gives their finite Ramanujan expansions and (9) in §3.

We introduce, now, another possible approach, in the study of f, g correlations.

In fact, apart from these *finite* expansions (even if depending on N, h) that we introduced in [8] (see §3) which are relative to the single and arbitrary functions f and g, we may consider the SHIFT-RAMANUJAN EXPANSION, or "Ramanujan expansion with respect to the shift", abbreviated s.R.e., of f, g correlation, namely

$$C_{f,g}(N,h) = \sum_{\ell=1}^{\infty} \widehat{C_{f,g}}(N,\ell) c_\ell(h) \quad (5)$$

where now the main issue is the possibility to give such an expansion, with some "SHIFT-RAMANUJAN COEFFICIENTS", $\widehat{C_{f,g}}(N,\ell)$, and whether we have in (5) an absolutely or uniformly convergent series. Many classical results in the literature, like our results above for $C_{f,g}(N,h)$, are all pointing towards the heuristic formula for these coefficients

$$\widehat{C_{f,g}}(N,\ell) \sim \widehat{f}(\ell)\widehat{g}(\ell)N \quad (6)$$

where the "$\sim$" sign is used like for Fourier coefficients formulæ, i.e., after suitable analytic assumptions and also with a well-specified analytic meaning.

The analytic assumptions ensuring "good" convergence may be very complicated, for these shift-Ramanujan expansions. However, the above for s.R.e. coefficients look like well-known heuristic formulæ, starting with the Hardy–Littlewood conjecture on $2k$-prime twins [17] (see §5).

In the following, "f is essentially bounded", i.e. $f(n) \ll_\varepsilon n^\varepsilon$, is tantamount to "$f$ satisfies the Ramanujan Conjecture". Hereafter, "$\forall \varepsilon > 0$", as usual, is implicit in bounds; in fact, $\varepsilon > 0$ is arbitrarily small and may change even in the same formula.

We give, inspired by these heuristics, the following general bounds, for all real $\delta > 0$: we say, by definition, that a "s.R.e. is in δ-class", whenever, for essentially bounded f, g,

$$\widehat{C_{f,g}}(N,\ell) \ll \frac{N^{1+\varepsilon}}{\ell^{1+\delta}},$$

with the implied constant depending eventually on both δ, ε. The noteworthy case $\delta = 1$ will be referred to as "s.R.e. is in the first class". For example, equation (6) above implies, by the bounds on f, g Ramanujan coefficients of (11) in §3, that our s.R.e. is in the first class (for the essentially bounded f, g and shift $h \ll N$, assuming also remainders in (6) are small enough). This notion will be useful at the end of §9 (and in future papers).

We define a *pure* Ramanujan expansion by $F(v) = \sum_{q=1}^{\infty} \widehat{F}(q)c_q(v)$, pointwise converging in v and in which the v-dependence is only in $c_q(v)$. For example, take [20], p. 24, in which both $\widehat{F}(q) = 1/q$ and $\widehat{F}(q) = 0$ (on all q) represent the constant zero function. On the other hand, Hildebrand's finite Ramanujan expansions (1.4), p. 167 of [23] are not pure in our sense.

Our main result is the following. Recall *Euler's function* is $\varphi(\ell) \stackrel{def}{=} |\{n \le \ell : (n, \ell) = 1\}|$.

Theorem 1. *Let $N, h \in \mathbb{N}$ and assume that $f, g : \mathbb{N} \to \mathbb{C}$ are essentially bounded, with, say, the finite* $\max\{q \ge 1 : g'(q) \ne 0\}$ *and f, g not depending on h. Consider the shift-Ramanujan expansion (5), abbreviated s.R.e., assuming, as we may,*[2] *that it converges pointwise, for all the fixed $h \in \mathbb{N}$. (We don't assume uniqueness of this expansion: we may even have undetermined coefficients.) Then, the following are equivalent.*

(*i*) *The s.R.e. is uniformly convergent (i.e., (17) in §5) and pure;*
(*ii*) *the s.R.e. coefficients are given by Carmichael's formula*

$$\widehat{C_{f,g}}(N, \ell) = \frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \sum_{h \le x} C_{f,g}(N, h) c_\ell(h);$$

(*iii*) *the s.R.e. coefficients are given by the explicit formula*

$$\widehat{C_{f,g}}(N, \ell) = \frac{\widehat{g}(\ell)}{\varphi(\ell)} \sum_{n \le N} f(n) c_\ell(n);$$

(*iv*) *the s.R.e. is finite and pure.*

We will call a s.R.e. satisfying one (hence, all) of previous equivalent conditions a *regular* shift-Ramanujan expansion.

An example of a regular s.R.e. is the main term in (2), since $\widehat{f}(q) = 0$, $\forall q > N$, by (9) (compare next section's final remarks on the singular series as a sum).

However, there are examples of irregular s.R.e., one of which (an arithmetic function f_H depending on the parameter H) is given in §9. (For it we apply the concept of δ-class.)

We prove in §5 the following important consequence of our Theorem 1.

[2] By Hildebrand's Theorem, see §1 end in [20] and reference [10] therein, also, compare beginning of §5.

Corollary 1. *With the hypotheses of Theorem 1, when the s.R.e. is regular and the f Eratosthenes transform f' is supported up to D (i.e., $f'(d) = 0, \forall d > D$), with $D \ll N$ as $N \to \infty$, we get*

$$C_{f,g}(N, h) = \mathfrak{S}_{f,g}(h)N + O_{\varepsilon}\left(N^{\varepsilon} D\right).$$

In particular, when $\frac{\log D}{\log N} < 1$, say $\frac{\log D}{\log N} < 1 - \delta$ for a certain small $\delta > 0$, we have

$$C_{f,g}(N, h) = \mathfrak{S}_{f,g}(h)N + O\left(N^{1-\delta}\right).$$

The last section is devoted to the lemmas that have both an application elsewhere and are of independent interest.

The paper is organized as follows. First of all, the next section expresses one main idea originating in the previous paper of this series, relating the arithmetic functions f and g with their truncated divisor sums counterparts, inside their correlation $C_{f,g}$. In fact, $C_{f,g}$ is unaltered by this process of divisors truncation: this vital remark started the study of *finite* Ramanujan expansions in [8].

In the same section, we then study a heuristic formula for correlations, based on leaving the fractional parts as not belonging to “main terms” (see Proposition 1). The next section (§3) transforms the truncated divisor sums into finite Ramanujan expansions and highlights a kind of duality between the two, through the formulæ, transforming Eratosthenes transforms into Ramanujan coefficients and vice versa.

It is very natural, then, to give examples of this new kind of Ramanujan expansions, the finite ones, in §4; these are obtained truncating the divisor ranges of given classic arithmetic functions (like the von Mangoldt Λ into Λ_N, with divisors up to N).

Actually, we will make a comparison between the given $f : \mathbb{N} \to \mathbb{C}$ and its truncation f_D, having divisors up to D, choosing $f = \Lambda$ and three specific examples of multiplicative functions (two of which are from our paper [7]).

After this short parade of formulæ (also applied in specific examples of finite Ramanujan expansions), starting the construction of a general theoretic framework for finite expansions of general arithmetic functions, we introduce in §5 a very special kind of expansion: the one for correlations $C_{f,g}(N, h)$, with respect to the shift $h \in \mathbb{N}$. These new *shift Ramanujan expansions* are, of course, linked to the two single, finite Ramanujan expansions of f and g: all of this shows up, clearly, in the proof of Theorem 1 that we provide in §5 together with the proof of Corollary 1.

In this section we show, also, both classic singular series and, say, “singular sums” (which are finite, due to finite Ramanujan expansions of f & g) as a result coming naturally out of the shift Ramanujan expansion and out of heuristics already encountered in §2. In fact, we explicitly give the classic singular series for twin primes (from $C_{\Lambda,\Lambda}$, the autocorrelation of von Mangoldt Λ) and the related singular sum (from C_{Λ_N,Λ_N}, the autocorrelation of N-truncated divisor sum Λ_N for Λ). The other two examples are not reported, due to space and complexity reasons, but are easily obtainable from the two multiplicative functions studied in §4 we quoted before (coming from our paper [7], see §4 second and third examples).

The truncated divisor sums f, in general, assuming that f satisfies the Ramanujan conjecture, are called *sieve functions*, see §6, and have been studied by the first author; in this section they are proposed to show how an immediate approach based on the large sieve is unable to give good bounds for remainder terms, in the asymptotic formula for correlations (compare our Corollary 1): this elementary approach is the object of Proposition 2. This, in turn, has also an elementary proof based on the elementary bound $O(1)$ for fractional parts.

However, this approach is very limited for its nature, not for the technology we may apply to it; in fact, even the most sophisticated bounds on bilinear forms of Kloosterman fractions (see [5]) give a bound that is very far, from the remainders well below N, in correlations' asymptotic formulæ (compare heuristics, in §2). This provides evidence for the necessity of a complete study of shift-Ramanujan expansions, as a powerful new method; hopefully, strong enough to produce much better error terms (for suitable arithmetic functions f and g).

In §7, we discuss how sieving can enhance the estimations.

Concluding remarks and future directions are in our penultimate §8.

Our last section §9 is an Appendix, containing (a fact and) Lemmas which we apply, but have an interest of their own.

2. Shifted convolution sums and truncated divisor sums

The heuristics for our correlations (compare classic [17]: eq. (5.26) and Conjecture B, with papers [7,8,11,21]) are of the kind

$$C_{f,g}(N,h) = \mathfrak{S}_{f,g}(h)N + \text{good remainder} \tag{7}$$

(say, $O(N^{1-\delta})$, for a $\delta > 0$), the singular series $\mathfrak{S}_{f,g}(h)$ for f and g, of shift $h \geq 0$, being defined as above. A justification for this heuristic comes from the following considerations.

For any $f, g : \mathbb{N} \to \mathbb{C}$ we defined Eratosthenes transforms f' and g', so $f(n) = \sum_{d|n} f'(d)$ and $g(m) = \sum_{q|m} g'(q)$; from our previous paper [8] we know that, for our purposes (namely, confining to $C_{f,g}$ study), f, g may be truncated over the divisors, as in (4), getting

$$f(n) = \sum_{d|n, d\leq N} f'(d) \qquad \text{and} \qquad g(m) = \sum_{q|m, q\leq N+h} g'(q). \tag{8}$$

Thus in studying $C_{f,g}$ we naturally find the *finite Ramanujan expansions* of f and g (see (9) in the next section).

We use these truncated divisor sum representations for f and g to deduce a "toy result":

Proposition 1. *For any $f, g : \mathbb{N} \to \mathbb{C}$ we have*

$$C_{f,g}(N,h) = \mathfrak{S}_{f,g}(h)N + O\left(\sum_{d \le N} |f'(d)| \sum_{q \le N+h} |g'(q)|\right).$$

Proof. In fact,

$$C_{f,g}(N,h) = \sum_{d \le N} f'(d) \sum_{q \le N+h} g'(q) \sum_{\substack{n \le N \\ n \equiv 0 \bmod d \\ n \equiv -h \bmod q}} 1$$

$$= \sum_{l|h} \sum_{d \le N} f'(d) \sum_{\substack{q \le N+h \\ (q,d)=l}} g'(q) \sum_{\substack{n \le N \\ n \equiv 0 \bmod d \\ n \equiv -h \bmod q}} 1 = \sum_{\substack{l|h \\ b:=-h/l}} \sum_{t \le \frac{N}{l}} f'(lt) \sum_{\substack{r \le \frac{N+h}{l} \\ (r,t)=1}} g'(lr) \sum_{\substack{m \le \frac{N}{lt} \\ m \equiv \bar{t} b \bmod r}} 1,$$

defining $\bar{t} \bmod r$ with $\bar{t}t \equiv 1 \bmod r$. An approach based on writing

$$\sum_{\substack{m \le \frac{N}{lt} \\ m \equiv \bar{t} b \bmod r}} 1 = \frac{N}{ltr} + O(1) = \frac{Nl}{(lt)(lr)} + O(1),$$

since the fractional parts are bounded at once, gives the error that is "non-optimal", i.e.

$$\ll \sum_{l|h} \sum_{t \le \frac{N}{l}} |f'(lt)| \sum_{\substack{r \le \frac{N+h}{l} \\ (r,t)=1}} |g'(lr)| \ll \sum_{l|h} \sum_{d \le N} |f'(d)| \sum_{\substack{q \le N+h \\ (q,d)=l}} |g'(q)|$$

$$\ll \sum_{d \le N} |f'(d)| \sum_{q \le N+h} |g'(q)|$$

and, say, as main term, recalling that we are truncating f' at N and g' at $N+h$,

$$\sum_{l|h} \sum_{d \le N} f'(d) \sum_{\substack{q \le N+h \\ (q,d)=l}} g'(q) \frac{Nl}{dq} = N \sum_{l|h} l \sum_{d \le N} \frac{f'(d)}{d} \sum_{\substack{q \le N+h \\ (q,d)=l}} \frac{g'(q)}{q}$$

$$= N \sum_{l|h} l \sum_{d} \frac{f'(d)}{d} \sum_{\substack{q \\ (q,d)=l}} \frac{g'(q)}{q},$$

which is also the main term in the heuristic (7), since Lemma A.6 in §9 gives

$$\mathfrak{S}_{f,g}(h) = \sum_{l|h} l \sum_{d} \frac{f'(d)}{d} \sum_{\substack{q \\ (q,d)=l}} \frac{g'(q)}{q}. \qquad \square$$

Remark 1. The same fractional parts, after Fourier expansion and an application of a result of Duke, Friedlander & Iwaniec, give (for similar correlations) the non-trivial results in [5].

In these formulæ, for the singular series, we can manage the series exchanges easily, from finiteness (compare Lemma A.6 proof, in §9). The same singular series, in fact, is simply a "*singular sum*". This feature, like finiteness of Ramanujan expansions involved, seems to have been overlooked in the literature. Actually, our *singular sum* may be seen as the N-th partial sum, of the original singular series. (We leave, as an exercise for the interested reader, to prove that the tail converges very rapidly to zero, as $N \to \infty$.)

3. The finite Ramanujan expansions: properties and formulæ

The truncated divisor sums for f and g, in (8), have (compare [8] Introduction) finite Ramanujan expansions, that we will sometimes abbreviate f.R.e.:

$$f(n) = \sum_{r \le N} \widehat{f}(r) c_r(n) \qquad \text{and} \qquad g(m) = \sum_{s \le N+h} \widehat{g}(s) c_s(m), \tag{9}$$

with an *explicit formula for* their *Ramanujan coefficients* that we proved in [8] Introduction (soon after Lemma 1):

$$\widehat{f}(r) = \sum_{\substack{m \\ m \equiv 0 \bmod r}} \frac{f'(m)}{m} = \frac{1}{r} \sum_{n \le \frac{N}{r}} \frac{f'(rn)}{n}, \quad \widehat{g}(s) = \sum_{\substack{m \\ m \equiv 0 \bmod s}} \frac{g'(m)}{m} = \frac{1}{s} \sum_{n \le \frac{N+h}{s}} \frac{g'(sn)}{n}. \tag{10}$$

Notice that this formula implies all truncated divisor sums have finite Ramanujan expansions.

In particular, for the essentially bounded f, g, we get the bounds

$$\widehat{f}(r) \ll_{\varepsilon} \frac{N^{\varepsilon}}{r}, \quad \widehat{g}(s) \ll_{\varepsilon} \frac{(N+h)^{\varepsilon}}{s}. \tag{11}$$

The other *explicit formula*, this time *for the Eratosthenes transform* in terms of Ramanujan coefficients (see the Introduction of [8], soon before Theorem 1), is:

$$f'(d) = d \sum_{j=1}^{\infty} \mu(j) \widehat{f}(dj) = d \sum_{j \le \frac{N}{d}} \mu(j) \widehat{f}(dj), \;\; g'(q) = q \sum_{j=1}^{\infty} \mu(j) \widehat{g}(qj) = q \sum_{j \le \frac{N+h}{q}} \mu(j) \widehat{g}(qj).$$

We profit to prove it briefly: apply (10), then the well-known Möbius inversion formula $\sum_{j|K} \mu(j) = [1/K]$ (with [ ] the integer part),

$$d \sum_{j \le \frac{N}{d}} \mu(j) \widehat{f}(dj) = \sum_{j \le \frac{N}{d}} \frac{\mu(j)}{j} \sum_{n \le \frac{N}{dj}} \frac{f'(djn)}{n} = \sum_{K \le \frac{N}{d}} \frac{f'(dK)}{K} \sum_{j|K} \mu(j) = f'(d).$$

These formulæ link Ramanujan coefficients (resp., $\widehat{f}$, $\widehat{g}$), with Eratosthenes transforms (resp., f', g'). This is a kind of duality: truncated divisor sums (with f', g') and finite

Ramanujan expansions (with $\widehat{f}, \widehat{g}$) clearly describe the same objects (our functions f, g). Furthermore, for the *high* Ramanujan coefficients, i.e., having index $Q/2 < q \le Q$, when divisors are truncated at Q, i.e., for Eratosthenes transform support $\subseteq [1, Q]$, we have:

$$u(n) = \sum_{d|n, d\le Q} u'(d) \quad \Longrightarrow \quad \widehat{u}(q) = u'(q)/q, \ \forall q \in (Q/2, Q],$$

entailing

$$\widehat{f}(r) = \frac{f'(r)}{r}, \ \forall r \in \left(\frac{N}{2}, N\right] \quad \text{and} \quad \widehat{g}(s) = \frac{g'(s)}{s}, \forall s \in \left(\frac{N+h}{2}, N+h\right]. \tag{12}$$

4. The finite Ramanujan expansions: examples

These two formulæ in (12) immediately imply for the von Mangoldt function, say,

$$\Lambda_N(n) = \sum_{d|n, d\le N} (-\mu(d)\log d),$$

which has been truncated as above for the calculation of $C_{\Lambda,\Lambda}(N, h)$,

$$\widehat{\Lambda_N}(r) = -\frac{\mu(r)\log r}{r}, \ \forall r \in \left(\frac{N}{2}, N\right], \widehat{\Lambda_{N+h}}(s) = -\frac{\mu(s)\log s}{s}, \ \forall s \in \left(\frac{N+h}{2}, N+h\right]. \tag{13}$$

More precisely,

$$C_{\Lambda,\Lambda}(N,h) = \sum_{n\le N} \Lambda(n)\Lambda(n+h) = \sum_{d\le N} \mu(d)(\log d) \sum_{q\le N+h} \mu(q)(\log q) \sum_{\substack{n\le N \\ n\equiv 0 \bmod d \\ n+h\equiv 0 \bmod q}} 1$$

$$= \sum_{d\le N} \mu(d)(\log d) \sum_{q\le N} \mu(q)(\log q) \sum_{\substack{n\le N \\ n\equiv 0 \bmod d \\ n+h\equiv 0 \bmod q}} 1 - \sum_{N<q\le N+h} \mu(q)(\log q) \sum_{\substack{n\le N \\ n+h\equiv 0 \bmod q}} \Lambda(n)$$

$$= C_{\Lambda,\Lambda_N}(N,h) + O\left(\log^2(N+h) \sum_{N<q\le N+h} \left(\frac{N}{q} + 1\right)\right).$$

Remainder terms are clearly $\ll h\log^2(N+h)$, which, if $h = o(N)$ is small enough, say $h \ll N^{1-\delta}$ (for a fixed $\delta > 0$), are negligible.

Hence we may stick to only one truncation, the one with N (ignoring the shift). Of course, for h small enough, this procedure works for all arithmetic functions f, g that do not grow too fast, like the ones satisfying (like Λ) the Ramanujan Conjecture: $f(n), g(n) \ll n^{\varepsilon}$.

Since all of our examples will satisfy this growth condition (like all interesting arithmetic functions, otherwise we may re-normalize) we get

$$C_{f,g_N}(N,h) \overset{def}{=} \sum_{d\le N} f'(d) \sum_{q\le N} g'(q) \sum_{\substack{n\le N \\ n\equiv 0 \bmod d \\ n+h\equiv 0 \bmod q}} 1 = C_{f,g}(N,h) + O_\varepsilon\left(N^\varepsilon (N+h)^\varepsilon h\right),$$

that is, all remainder terms are negligible (esp., $h \ll N^{1-\delta}$, for a fixed $\delta > 0$), so we may set a common truncation for both f and g:

$$f(n) = \sum_{d|n, d\le N} f'(d) \qquad \text{and} \qquad g(m) = \sum_{q|m, q\le N} g'(q).$$

Returning to our von Mangoldt function $f = g = \Lambda$ (hence, $C_{f,g}(N,h)$ regards h-twins of primes), the idea of truncating its divisors has been pursued by many authors in the literature (mainly, in the area of sieve methods), but specifically by Goldston & Yıldırım [15]: in the 2000s, has given spectacular consequences at the hands of Goldston, Pintz & Yıldırım [12–14], which have been applied also by Green & Tao [16] (to prove that the sequence of primes contains arbitrary long arithmetic progressions) and, more recently, by Maynard [18] and others, like the Polymath project (see the web), to study bounded gaps between primes.

We wish to emphasize that such an approach has not yet been followed, in order to give hints in the asymptotic formulæ, for the correlation sum of twin primes (say, $C_{\Lambda,\Lambda}$ here)! Thus, with (13) above, we try to give a new flavor to the estimate of Ramanujan coefficients of Λ; these high coefficients are somehow unexpected as they do not agree, exactly, with the known classical ones.

However, our formulæ, specifically (10), give for the coefficients

$$\widehat{\Lambda_N}(q) = -\frac{1}{q} \sum_{n\le \frac{N}{q}} \frac{\mu(qn)\log(qn)}{n} = -\frac{\mu(q)\log q}{q} \sum_{\substack{n\le \frac{N}{q} \\ (n,q)=1}} \frac{\mu(n)}{n} - \frac{\mu(q)}{q} \sum_{\substack{n\le \frac{N}{q} \\ (n,q)=1}} \frac{\mu(n)\log n}{n} \tag{14}$$

which, heuristically speaking, for q small with respect to N, are in good agreement with the classical known formulæ, i.e.

$$\sim \frac{\mu(q)}{\varphi(q)},$$

from the very well-known formulæ, see [2] or [19], for $q \le x$ (so, for $q \le \sqrt{N}$ in our case), $c > 0$ fixed:

$$\sum_{\substack{n\le x \\ (n,q)=1}} \frac{\mu(n)}{n} \ll \exp\left(-c\sqrt{\log x}\right), \qquad \sum_{\substack{n\le x \\ (n,q)=1}} \frac{\mu(n)\log n}{n} = -\frac{q}{\varphi(q)} + O\left(\exp\left(-c\sqrt{\log x}\right)\right). \tag{15}$$

The fact that we are working with f.R.e. (equivalently, of truncating divisors) makes the coefficients behave in a different way, with respect to the classical Ramanujan (series) expansions; in particular, the "low" coefficients (i.e., with $q \le \sqrt{N}$ in above example) should agree to some extent with the classical ones, while the effect of truncating divisors is clear on the last ones (we call "high", see the above), which may be totally different!

We consider now three other examples, the first two of which are related to [7], respectively to Corollary 1 of [7] and to Corollary 2 of [7]. Our third of these (and last example) will be related to these two (but we didn't mention it in our earlier papers).

Our next example comes from the arithmetic function, say (see Corollary 1 [7]),

$$f_s(n) = \frac{\sigma_s(n)}{n^s} = \frac{1}{n^s}\sum_{d|n} d^s = \sum_{d|n}\left(\frac{n}{d}\right)^{-s} = \sum_{d|n} d^{-s} = \sigma_{-s}(n),$$

by passing from $d|n$ to its complementary divisor $\frac{n}{d}\,\big|\, n$. For this function we have, for all $s > 0$,

$$\sigma_{-s}(n) = \frac{\sigma_s(n)}{n^s} = \sum_{q=1}^{\infty} \widehat{\sigma_{-s}}(q) c_q(n)$$

as a classical Ramanujan expansion (even *converging absolutely*, by Lemma A.1 in §9).

Here for notational convenience we write $f_s(n) := \frac{\sigma_s(n)}{n^s} = \sigma_{-s}(n)$, introducing

$$f_s(n) = \sum_{d|n} f'_s(d), \quad \text{with } f'_s \stackrel{def}{=} f_s * \mu \quad \Longrightarrow \quad f_{s,D}(n) \stackrel{def}{=} \sum_{\substack{d|n \\ d\le D}} f'_s(d),$$

(for fixed $s > 0$ and $D \in \mathbb{N}$) its *truncated* counterpart, *over the divisors* up to D. This definition is pretty general; here, in the present example,

$$f'_s(d) = d^{-s} \quad \Longrightarrow \quad \sigma_{-s,D}(n) \stackrel{def}{=} \sum_{\substack{d|n \\ d\le D}} d^{-s} = \sum_{q\le D} \widehat{\sigma_{-s,D}}(q) c_q(n)$$

(for all $s > 0$ and $D \in \mathbb{N}$, both fixed), with finite Ramanujan coefficients

$$\widehat{\sigma_{-s,D}}(q) \stackrel{def}{=} \sum_{\substack{m\le D \\ m\equiv 0 \bmod q}} m^{-s-1} = \frac{1}{q^{s+1}} \sum_{n\le \frac{D}{q}} \frac{1}{n^{s+1}}.$$

Notice that these are different from the classical Ramanujan coefficients, that we calculated thanks to the Delange 1976 [10] Theorem (see Theorem 3 and following discussion, before Theorem 4, in [20])

$$\widehat{\sigma_{-s}}(q) \stackrel{def}{=} \sum_{m\equiv 0 \bmod q} m^{-s-1} = \frac{1}{q^{s+1}} \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} = \frac{1}{q^{s+1}} \zeta(s+1)$$

(apart from similarity we'll check soon, for suitable indices), since the finite ones have

$$\frac{D}{2} < q \le D \;\Rightarrow\; \widehat{\sigma_{-s,D}}(q) = \frac{1}{q^{s+1}},$$

as we already knew from (12), in the previous §3. (Also, these trivially vanish for $q > D$, while the classical ones don't.)

However, if the indices are somehow "small", the f.R.e. has coefficients that are asymptotic to classical ones, as we see now: when $D \to \infty$,

$$\widehat{\sigma_{-s,D}}(q) = \frac{1}{q^{s+1}} \sum_{n \le \frac{D}{q}} \frac{1}{n^{s+1}} = \frac{\zeta(s+1)}{q^{s+1}} - \frac{1}{q^{s+1}} \sum_{n > \frac{D}{q}} \frac{1}{n^{s+1}} = \frac{\zeta(s+1)}{q^{s+1}} \left(1 + O_s\left(\left(\frac{q}{D}\right)^s\right)\right),$$

namely

$$q = o(D) \;\Rightarrow\; \widehat{\sigma_{-s,D}}(q) = \widehat{\sigma_{-s}}(q)\,(1 + o_s(1)) \sim \widehat{\sigma_{-s}}(q).$$

Thus the finite Ramanujan coefficients, on "small indices" (say, $q = o(D)$ here) are asymptotic to the Ramanujan coefficients (of the "classical" expansion).

Furthermore, with $f_s(n) = \prod_{p|n} (1 - p^{-s})$, $s > 0$, we get Corollary 2 [7] application and, from multiplicativity,

$$f_s(n) = \sum_{d|n} \mu(d) d^{-s} \;\Rightarrow\; f_s'(d) = \mu(d) d^{-s},$$

giving rise again to

$$q = o(D) \;\Rightarrow\; \widehat{f_{s,D}}(q) = \widehat{f_s}(q)\,(1 + o_s(1)) \sim \widehat{f_s}(q), \tag{16}$$

by the same calculations as above.

This suggests the third example of this kind, i.e., say

$$f_s(n) = \prod_{p|n} \left(1 + p^{-s}\right) = \sum_{d|n} \mu^2(d) d^{-s} \;\Rightarrow\; f_s'(d) = \mu^2(d) d^{-s}$$

with this kind of truncation

$$f_{s,D}(n) = \sum_{\substack{d|n \\ d \le D}} \mu^2(d) d^{-s} \;\Rightarrow\; f_{s,D}(n) = \sum_{q \le D} \widehat{f_{s,D}}(q) c_q(n)$$

with the same behavior as given in (16).

More generally, Delange's Theorem [10] gives (compare Theorem 3 in [20])

$$\widehat{f}(q) = \sum_{m \equiv 0 \bmod q} \frac{f'(m)}{m},$$

with the hypothesis

$$\sum_{m=1}^{\infty} \frac{2^{\omega(m)}|f'(m)|}{m} < \infty,$$

which is certainly satisfied by the D-truncation of our f (since we have a f.R.e. for it), say

$$f_D(n) = \sum_{\substack{d|n \\ d \le D}} f'(d) \;\Rightarrow\; \widehat{f_D}(q) = \sum_{\substack{m \le D \\ m \equiv 0 \bmod q}} \frac{f'(m)}{m},$$

which we proved directly (actually, the same method, but applying analytic approximations, too, of course, proves Delange's Theorem, compare [20]). We wish to have $\widehat{f_D}(q) \sim \widehat{f}(q)$.

For this, the only problem is the effect of truncation on Ramanujan coefficients, that is,

$$\text{differences} \sum_{\substack{m > D \\ m \equiv 0 \bmod q}} \frac{f'(m)}{m} = \frac{1}{q} \sum_{n > D/q} \frac{f'(qn)}{n} \text{ must be infinitesimal}$$

and this is possible, clearly, only when the variable $D/q \to \infty$. That is, $q = o(D)$ here is a necessary condition; actually, *for the previous three cases* a *sufficient* one *too*. For the high coefficients we already observed a neat difference with the classical ones; this may be justified by the divisors' truncation itself, that has to change "last", so to speak, coefficients, in order to cope with the infinite tail, that is missing (of course, in *finite* Ramanujan expansions).

5. Ramanujan expansions with respect to the shift

In the following, we will dwell mainly with the easiest possible hypothesis for the series in (5), namely, the uniform convergence (i.e., not depending on the shift h)

$$\sum_{\ell=1}^{\infty} \widehat{C_{f,g}}(N,\ell) c_\ell(h) \text{ converges uniformly } \forall h \in \mathbb{N}. \tag{17}$$

We explicitly point out that we are considering series back again, since our previous remark, truncating the divisor sums (hence giving f.R.e. above), gives no hint on whether the present expansion in (17) is finite or not.

However, in the case of s.R.e. regularity (see Theorem 1) the shift expansion is finite. Our main problem is to try to understand *when* we have such regularity.

Even in the case of irregularity, we always have the pointwise convergence of our s.R.e., as an easy consequence of Hildebrand's Theorem (see Theorem 1 footnote in §1), for any arithmetic function: here, we want to study the f, g correlation as a function of the shift $h \in \mathbb{N}$; the problem, however, is that we don't have, a priori, the uniqueness for the s.R.e. and this may lead even to more different coefficients for the same expansion. So, uniform convergence of our s.R.e. confirms to be the easiest analytic assumption, especially in the light of Theorem 1.

We prove first Theorem 1 and, then, the much easier Corollary 1.

Proof of Theorem 1. We follow the loop: $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$, $(iii) \Rightarrow (iv)$, $(iv) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Apply Lemma A.4 in §9 to the arithmetic function $F(h) = C_{f,g}(N,h)$.

$(ii) \Rightarrow (iii)$. Expanding in finite Ramanujan expansion g, with a finite support of g' (hence of $\widehat{g}$) which does not depend on h (likewise for $f, \widehat{g}$), inside $C_{f,g}(N,h)$ (so may exchange h-sum) we get

$$\frac{1}{x}\sum_{h\le x} C_{f,g}(N,h)c_\ell(h) = \sum_q \widehat{g}(q)\sum_{n\le N} f(n)\frac{1}{x}\sum_{h\le x} c_q(n+h)c_\ell(h).$$

Passing to the limit and normalizing by Euler's function $\varphi(\ell)$ and applying (ii),

$$\widehat{C_{f,g}}(N,\ell) = \frac{1}{\varphi(\ell)}\sum_q \widehat{g}(q)\sum_{n\le N} f(n)\lim_{x\to\infty}\frac{1}{x}\sum_{h\le x} c_q(n+h)c_\ell(h),$$

in which, writing $\mathbf{1}_\wp \overset{def}{=} 1$ if $\wp$ is true, $\overset{def}{=} 0$ otherwise and $j \in \mathbb{Z}_\ell^*$ to abbreviate $j \le \ell$, $(j,\ell) = 1$, we have

$$\frac{1}{x}\sum_{h\le x} c_q(n+h)c_\ell(h) = \frac{1}{x}\sum_{r\in\mathbb{Z}_q^*} e^{2\pi i n r/q}\sum_{j\in\mathbb{Z}_\ell^*}\sum_{h\le x} e^{2\pi i(r/q-j/\ell)h} =$$

$$= \mathbf{1}_{q=\ell}c_\ell(n) + O\left(\frac{1}{x}\sum_{r\in\mathbb{Z}_q^*}\sum_{j\in\mathbb{Z}_\ell^*}(1-\mathbf{1}_{q=\ell}\mathbf{1}_{r=j})\frac{1}{\|r/q-j/\ell\|}\right) = \mathbf{1}_{q=\ell}c_\ell(n) + o(1),$$

letting: $x \to \infty$, proves the orthogonality relations (discussed in [20] with more details)

$$\lim_{x\to\infty}\frac{1}{x}\sum_{h\le x} c_q(n+h)c_\ell(h) = \mathbf{1}_{q=\ell}c_\ell(n),$$

giving at once (iii).

$(iii) \Rightarrow (iv)$. Observe that the support of $\widehat{g}$ is finite and independent of h.

$(iv) \Rightarrow (i)$. Trivial (since uniform convergence follows by h-independence and finiteness). This completes the proof. □

Proof of Corollary 1. We only need to prove the first formula, for which Lemma A.3 gives

$$\sum_{n\le N} f(n)c_\ell(n) = \widehat{f}(\ell)\varphi(\ell)N + O_\varepsilon\left((D\ell)^{1+\varepsilon}\right),$$

whence, from the explicit formula in Theorem 1 (recall $\ell, h, D \ll N$) again by Lemma A.1,

$$C_{f,g}(N,h) = \sum_{\ell\ll N} \frac{\widehat{g}(\ell)}{\varphi(\ell)}\left(\widehat{f}(\ell)\varphi(\ell)N\right)c_\ell(h) + O_\varepsilon\left(N^\varepsilon \sum_{\ell\ll N}\frac{D}{\ell}(\ell,h)\right)$$

$$= \sum_\ell \widehat{f}(\ell)\widehat{g}(\ell)c_\ell(h)N + O_\varepsilon\Big(N^\varepsilon D\sum_{t|h}\sum_{v\ll N}\frac{1}{v}\Big) = \mathfrak{S}_{f,g}(h)N + O_\varepsilon\Big(N^\varepsilon D\Big).$$

This completes the proof. □

The shift-Ramanujan expansion, for any pair of arithmetic functions f and g, leads us to a kind of entanglement of the two single Ramanujan expansions for f and g. Our heuristic formulæ, with those in the literature, are a kind of squeezing on the diagonal, as obtained considering the same moduli $r = s$ in the single Ramanujan expansions with $\widehat{f}(r)$ and $\widehat{g}(s)$.

This “reduction on the diagonal”, say, is a consequence, for our results [21], [7], of the decay bounds for the single Ramanujan coefficients. However, as the δ-class definition for the decay of this time the shift-Ramanujan coefficients reveals, this kind of reduction may hold in more general hypotheses (compare Corollary 1), than the ones we applied, now and in our previous studies (like, esp., [11], [21], [7]). In particular, in Theorem 1, the possibility to apply Carmichael’s formula (implying (iii), the explicit formula for the shift-Ramanujan expansion) seems to be the easiest requirement; we hope this will shed some light on the possibility to prove, in suitable, new hypotheses, the heuristic formulæ, like (2), for our shifted convolution sums.

We wish, at this point, to conclude with three classical singular series, for f, g correlations, thus giving (compare Corollary 1) heuristic formulæ.

Of course, our first example is the case $f = g = \Lambda$ of $2k$-twin primes (there’s a misprint in [7] at page 702):

$$\mathfrak{S}_{\Lambda,\Lambda}(h) \overset{def}{=} \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} c_q(h),$$

letting $h = 2k$ to avoid h odd (trivial case, with vanishing series). For it, we compare with the singular sum we get in case of truncations, say, Λ_N, i.e.

$$\mathfrak{S}_{\Lambda_N,\Lambda_N}(h) \overset{def}{=} \sum_{q=1}^{\infty} \widehat{\Lambda_N}(q)^2 c_q(h) = \sum_{q\le N} \widehat{\Lambda_N}(q)^2 c_q(h),$$

that (see (14) and (15)) has, in the range $q \le \sqrt{N}$,

$$\widehat{\Lambda_N}(q) = \frac{\mu(q)}{\varphi(q)} + O\left(\frac{1}{q}\exp\left(-c\sqrt{\log N}\right)\right) \Rightarrow$$
$$\widehat{\Lambda_N}^2(q) = \frac{\mu^2(q)}{\varphi^2(q)} + O\left(\frac{1}{\varphi^2(q)}\exp\left(-c\sqrt{\log N}\right)\right)$$

rendering

$$\mathfrak{S}_{\Lambda_N,\Lambda_N}(h) = \sum_{q\le\sqrt{N}} \frac{\mu^2(q)}{\varphi^2(q)} c_q(h) + O\left(\exp\left(-c\sqrt{\log N}\right)\sum_{q\le\sqrt{N}}\frac{(q,h)}{\varphi^2(q)}\right)$$
$$+ O\left(\sum_{q>\sqrt{N}}\frac{(q,h)}{\varphi^2(q)}\right),$$

from Lemma A.1; thus we may approximate (with a changed $c > 0$) as

$$\mathfrak{S}_{\Lambda_N,\Lambda_N}(h) = \mathfrak{S}_{\Lambda,\Lambda}(h) + O_h\left(\exp\left(-c\sqrt{\log N}\right)\right) + O_h\left(\frac{\log^2 N}{\sqrt{N}}\right). \tag{18}$$

We have used here the trivial bound, compare [2], $\varphi(q) \gg q/\log q$, inside the estimates

$$\sum_{q\le\sqrt{N}}\frac{(q,h)}{\varphi^2(q)} \ll \sum_{d|h} d \sum_{\substack{q\le\sqrt{N}\\ q\equiv 0 \bmod d}} \frac{\log^2 q}{q^2} \ll (\log^2 N)\sum_{d|h} d^{-1}\sum_{n\le\frac{\sqrt{N}}{d}}\frac{1}{n^2} \ll_h \log^2 N$$

and

$$\sum_{q>\sqrt{N}}\frac{(q,h)}{\varphi^2(q)} \ll \sum_{d|h} d \sum_{\substack{q>\sqrt{N}\\ q\equiv 0 \bmod d}} \frac{\log^2 q}{q^2} \ll \sum_{d|h} d^{-1}\sum_{n>\frac{\sqrt{N}}{d}}\frac{\log^2 n + \log^2 d}{n^2}$$
$$\ll \sum_{d|h,d\le\sqrt{N}}\frac{1}{d}\left(d\frac{\log^2 N}{\sqrt{N}} + \sum_{n>N}\frac{\log^2 n}{n^2}\right) + \sum_{d|h,d>\sqrt{N}}\frac{\log^2 d}{d} \ll_h \frac{\log^2 N}{\sqrt{N}}.$$

Thus (18) proves that the "singular sum" well approximates the original singular series. This is also true in the case $f = g = \Lambda$ (for which we don't have absolute convergence of Λ Ramanujan expansion, see [7]). But when we also have the absolute convergence of the original Ramanujan expansions, of both f and g separately (so we are considering the single two Ramanujan expansions), we get an even better approximation (as the diligent reader may check, in previous two examples).

In fact, if we consider the other two examples (of [7] Corollaries) given in the previous §4, we see a very useful convergence of Ramanujan expansions. We refer to our previous paper for the expression of the corresponding singular series (which are too involved to quote, due to space reasons). As the diligent reader may check, the singular series of these two examples (see §4 end) converge even better than our previous estimates, so the difference between them and finite Ramanujan expansions counterparts behaves much better than what we saw in (18) (whose error terms depend on zero-free regions for the Riemann ζ function).

6. Sieve functions

We first recall the definition [4]: a *sieve function* $a : \mathbb{N} \to \mathbb{C}$ of range A (that is an unbounded parameter, depending on other variables, see the following) may be written as

$$a(n) \stackrel{def}{=} \sum_{d|n, d\le A} v(d),$$

where the arithmetic function v is *essentially bounded*, namely we recall

$$v(n) \ll_\varepsilon n^\varepsilon, \qquad \text{as } n \to \infty.$$

In other words, a sieve a of range A is an A-truncated divisor sum satisfying the Ramanujan Conjecture. In fact, "a is essentially bounded", by Möbius inversion, is equivalent to "v is essentially bounded".

Hence, we immediately get the *finite Ramanujan expansion* for a sieve a of range A

$$a(n) = \sum_{q\le A} \widehat{a}(q) c_q(n), \quad \text{where} \quad \widehat{a}(q) \stackrel{def}{=} \sum_{\substack{d\le A \\ d\equiv 0 \bmod q}} \frac{v(d)}{d},$$

with (at most) A terms.

Notice that for our general definition a sieve function does not always come from a process of truncation over the divisors, from a fixed arithmetic function f, namely cutting the support of its Eratosthenes transform f'. In this section our sieve functions, say a, are given, once we are given the arithmetic function v and the range A. Furthermore, the bounds we aim at in the present section are for comparison with the ones we get from our Theorem 1 & its Corollary 1; so, we identify a sieve function a by means of two bounds it satisfies: namely, v (see the above) is essentially bounded, so it (and its powers) will contribute N^ε, while the range A is a power of N (say, $0 < \frac{\log A}{\log N} < 1$) and so it contributes correspondingly to our bounds. Of course, we may also put, in our O-bounds, a direct dependence on our sieve functions, but this would not render them transparent in the comparison with trivial bounds.

Also, observe that truncated divisor sums like $a(n)$ above are periodic, with a period dividing, say, $M := [1, \dots, A] \stackrel{def}{=} \text{l.c.m.}(1, \dots, A)$, the least common multiple of all naturals up to A. To see this, note that $a(n) = a((n, M))$ so that $a(n + M) = a(n)$.

Already $[1,\ldots,A]$ grows exponentially with A, that is typically a power of N; and we may take $v(d)$ growing again exponentially. In all, the periodicity for our truncated divisor sum $a(n)$, even if bounding $a(n)$ itself, may supply an estimate growing (more than) exponentially with N, our main variable. This should be compared with our requirement of growing at most as arbitrarily small powers of N, given by the Ramanujan-type estimate. That's the reason why (and we apologize here for the terminology conflict) the "essential boundedness", even if apparently weaker than boundedness (from periodicity) is, actually, much stronger.

Sieve functions f always have a mean-value (i.e., $M(f) \stackrel{def}{=} \lim_x \frac{1}{x}\sum_{n\le x} f(n)$, [20]) and it is $\widehat{f}(1)$ (this by Wintner's 1943 Theorem: [20], Theorem 2). Also, the Dirichlet series of a sieve f of range Q is the product of the Riemann ζ-function and a Dirichlet polynomial with (at most) Q terms: compare [4] (soon after (1.3) equation).

From our considerations in §2, a general arithmetic function f may be seen as a kind of sieve function of range N (if we confine to its correlations). However, there is no point in considering the range N, as it holds for all arithmetic functions f. Notice that the parameter $Q \ll N$ is dependent on N; however, $Q \to \infty$, as $N \to \infty$ (avoiding trivialities).

Hereafter, we assume $f'(Q) \neq 0$ and define the *level*[3] of our f as $\lambda(f) \stackrel{def}{=} (\log Q)/(\log N)$. Notice the sensitivity of this definition to the main variable $N \to \infty$.

We give now a result stemming from the large sieve inequality (however, with an elementary alternative proof).

Proposition 2. *For any pair of sieve functions $a, b : \mathbb{N} \to \mathbb{C}$ of ranges, resp., $A \le B$,*

$$C_{a,b}(N,h) = \mathfrak{S}_{a,b}(h)N + O_\varepsilon\left((AB)^{1+\varepsilon}\right). \tag{19}$$

Proof. Expanding a of range A and b of range $B \ge A$ in finite Ramanujan expansions, we get

$$C_{a,b}(N,h) = \sum_{d\le A} \widehat{a}(d) \sum_{q\le B} \widehat{b}(q) \sum_{n\le N} c_d(n)c_q(n+h), \tag{20}$$

that we study by a modified large sieve inequality using the fact that Farey fractions $\frac{j}{q} \neq \frac{r}{d}$ are well-spaced in $[0,1]$:

$$\sum_{n\le N} c_d(n)c_q(n+h) = \mathbf{1}_{d=q} N c_q(h) + O\Big(\sum_{j\in\mathbb{Z}_q^*}\sum_{r\in\mathbb{Z}_d^*} \mathbf{1}_{\frac{j}{q}\neq\frac{r}{d}} \frac{1}{\|j/q - r/d\|}\Big)$$

(compare details in $(ii) \Rightarrow (iii)$ proof of Theorem 1) and the bounds on the Ramanujan coefficients, coming from (10) in §3:

[3] For a discussion on links between the present definition and the *level of distribution* of f *in arithmetic progressions*, compare [4].

$$\widehat{a}(d) \ll_\varepsilon A^\varepsilon/d \quad \text{and} \quad \widehat{b}(q) \ll_\varepsilon B^\varepsilon/q \tag{21}$$

together give (by Lemma 2 of [9] for our AB-spaced Farey fractions $\lambda_r := r/d$, $\lambda_s := j/q$)

$$C_{a,b}(N,h) = N \sum_{q\le A} \widehat{a}(q)\widehat{b}(q)c_q(h)$$

$$+ O_\varepsilon\Big(A^\varepsilon \sum_{d\le A} \frac{1}{d^2} \sum_{j\in\mathbb{Z}_d^*} \sum_{\substack{r\in\mathbb{Z}_d^* \\ r\ne j}} \frac{1}{\|\frac{j-r}{d}\|} + B^\varepsilon \sum_{d\le A} \sum_{\substack{q\le B \\ q\ne d}} \frac{1}{dq} \sum_{j\in\mathbb{Z}_q^*} \sum_{r\in\mathbb{Z}_d^*} \frac{1}{\|j/q - r/d\|}\Big)$$

$$= \mathfrak{S}_{a,b}(h)N + O_\varepsilon\Big(A^\varepsilon \sum_{d\le A} \frac{\varphi(d)}{d^2} \sum_{\Delta\le d/2} \frac{d}{\Delta} + AB^{1+\varepsilon}\Big(\sum_{d\le A} \frac{1}{d^2} \sum_{r\in\mathbb{Z}_d^*} 1\Big)^{1/2}\Big(\sum_{q\le B} \frac{1}{q^2} \sum_{j\in\mathbb{Z}_q^*} 1\Big)^{1/2}\Big),$$

whence

$$C_{a,b}(N,h) = \mathfrak{S}_{a,b}(h)N + O_\varepsilon\left(A^{1+\varepsilon} + (AB)^{1+\varepsilon}\right) = \mathfrak{S}_{a,b}(h)N + O_\varepsilon\left((AB)^{1+\varepsilon}\right). \quad \square$$

Sketch for an Alternative Proof. Instead of taking finite Ramanujan expansions, we can follow the argument of Proposition 1, say, to infer that "fractional parts are bounded". QED

Remark 2. If $AB \ll N^{1-\delta}$, for a fixed $\delta > 0$, then (19) is an asymptotic formula; i.e., calling $\lambda(a)$ and $\lambda(b)$, resp., the levels of, resp. a and b, the requirement is $\lambda(a)+\lambda(b) < 1$. In the particular case $a = b$ (a autocorrelation) this means $\lambda(a) < 1/2$ which is the well-known barrier for the large sieve technique (which we do not apply here but we rely actually on its proof as the main ingredient, namely the well-spaced property of Farey fractions). Notice the uniformity in $h \ge 0$ (uniformity with respect to the shift which is useful for the correlation asymptotic formulæ). Again we can compare Remark 1 and the better results in [5].

7. Sifting from small prime divisors

We give a new definition which will be useful, when applying our study to sieve functions that, in some sense (we specify now), have no divisors with "small primes".

We say that a general $f : \mathbb{N} \to \mathbb{C}$ satisfies a *sieve condition up to* G (> 1 and integer) when its Eratosthenes transform f' (recall, defined by $f * \mu$) has the property

$$p \le G,\ p|q \implies f'(q) = 0.$$

Defining, as usual, the product of all primes up to G as

$$P(G) \overset{def}{=} \prod_{p\le G} p,$$

notice that we obtain (for f with a sieve condition up to G) a kind of sifting f from small primes $p \leq G$, the formula

$$f(n) = f\left(\frac{n}{(n, P(G))}\right), \qquad \forall n \in \mathbb{N},$$

since $f(n) = \sum_{q|n} f'(q)$.

We call f a *G-sifted function* of range Q, whenever it is a sieve function of range Q and satisfies a sieve condition up to G.

Notice that, from the formula (10) for Ramanujan coefficients, we get that any G-sifted function f (of range Q) has a finite Ramanujan expansion without indices from 2 up to G:

$$f(n) = \widehat{f}(1) + \sum_{G<q\leq Q} \widehat{f}(q) c_q(n).$$

Also, the coefficients $\widehat{f}(q) = 0$ when q has a prime factor $p \leq G$. More precisely,

$$f \text{ is } G-\text{sifted of range } Q \implies f(n) = \widehat{f}(1) + \sum_{\substack{G<q\leq Q \\ (q,P(G))=1}} \widehat{f}(q) c_q(n).$$

In particular, we have that the singular series $\mathfrak{S}_f(h) = \mathfrak{S}_{f,f}(h)$ (taking $g = f$ for the heuristic of $C_f = C_{f,f}$), in case $h > 0$, of a G-sifted f of range Q has the shape (using (21) here)

$$\mathfrak{S}_f(h) = \widehat{f}(1)^2 + \sum_{\substack{G<q\leq Q \\ (q,P(G))=1}} \widehat{f}(q)^2 c_q(h)$$
$$= \widehat{f}(1)^2 + O_\varepsilon\left(Q^\varepsilon \sum_{G<q\leq Q} \frac{1}{q^2}|c_q(h)|\right) = \widehat{f}(1)^2 + O_\varepsilon\left(Q^\varepsilon \max_{G\leq A\ll Q} \frac{1}{A^2} \sum_{A<q\leq 2A} |c_q(h)|\right)$$
$$= \widehat{f}(1)^2 + O_\varepsilon\left(Q^\varepsilon d(h)/G\right) = \widehat{f}(1)^2 + O_\varepsilon\left((hQ)^\varepsilon/G\right),$$

applying a dyadic argument based on the following bound, for all integers $0 \leq A < B$, using $|c_q(h)| \leq (q, h)$, Lemma A.1 §9, here:

$$\sum_{A<q\leq B} |c_q(h)| \leq \sum_{l|h} l \sum_{\substack{A<q\leq B \\ (q,h)=l}} 1 \leq \sum_{l|h} l \sum_{\substack{A<q\leq B \\ q\equiv 0 \bmod l}} 1$$
$$= \sum_{l|h} l\left(\left[\frac{B}{l}\right] - \left[\frac{A}{l}\right]\right) \leq \sum_{l|h,\, l\leq B} l\left(\frac{B}{l} - \frac{A}{l} + 1\right) \leq 2\sum_{l|h} B \ll Bd(h).$$

Heuristically speaking, we have a kind of general philosophy, for sifted functions: “low”, say, Ramanujan coefficients vanish (like, on the other side, those out of range). The absence of low primes, $p \le G$, in conjunction with low shifts, again up to G, gives the following, interesting properties, like (22) & (23).

As an example, let's see what happens, for our sieve functions in arithmetic progressions.

Proposition 3. *For any sieve function $f : \mathbb{N} \to \mathbb{C}$ of range $D \ll N$, we have uniformly in the non-zero residue classes $h \in [-G, G]$, whenever the modulus t is G-sifted, $G = o(N)$,*

$$\sum_{\substack{n \le N \\ n \equiv -h \bmod t}} f(n) = \frac{N}{t}\widehat{f}(1) + O_{\varepsilon}\left(D^{\varepsilon}\left(\frac{Nt^{\varepsilon}}{tG} + D\right)\right). \tag{22}$$

Proof. Lemma A.2, §9, says that for a sieve f of range $D \ll N$ we have

$$\sum_{\substack{n \le N \\ n \equiv a \bmod t}} f(n) = \frac{N}{t}\sum_{k|t}\widehat{f}(k)c_k(a) + O_{\varepsilon}\left(D^{1+\varepsilon}\right);$$

so assume t is G-sifted (a classical expression to mean $(t, P(G)) = 1$, here) and say the shift $h = -a$ satisfies $0 < |a| = |h| \le G$: then, any prime divisor p of $k|t$ in the right hand side above has to be $p > G$, while any prime divisor of a cannot be greater than G itself. This formula simplifies to (using $\widehat{f}(k) \ll_{\varepsilon} D^{\varepsilon}/k$ and the fact that $k|t$ and $k > 1$ $\Rightarrow k > G$)

$$\sum_{\substack{n \le N \\ n \equiv a \bmod t}} f(n) = \frac{N}{t}\sum_{k|t}\widehat{f}(k)\mu(k) + O_{\varepsilon}\left(D^{1+\varepsilon}\right) = \frac{N}{t}\widehat{f}(1) + O_{\varepsilon}\left(D^{\varepsilon}\left(\frac{Nt^{\varepsilon}}{tG} + D\right)\right),$$

entailing (22), which is uniform in the non-zero residue classes $h \in [-G, G]$. □

Also, the same hypotheses give another interesting property.

Proposition 4. *The same hypotheses of Proposition 3 give*

$$\sum_{n \le N} f(n)c_t(n+h) = \mu(t)\widehat{f}(t)N + O_{\varepsilon}\left((Dt)^{1+\varepsilon}\right). \tag{23}$$

Proof. From Lemma A.3 of §9, choosing $a = -h$, uniformly in $0 < |a| \le G$, whenever $\ell = t$ is G-sifted, we have $(a, t) = 1 \Rightarrow c_t(a) = \mu(t)$, whence

$$\sum_{n \le N} f(n)c_t(n+h) = \mu(t)\widehat{f}(t)N + O_{\varepsilon}\left((Dt)^{1+\varepsilon}\right). \quad \square$$

These formulæ are very useful inside the correlations.

Proposition 5. *Take the sieve functions $f, g : \mathbb{N} \to \mathbb{C}$ of, resp., ranges $D, Q \ll N$ and assume g is G-sifted, where $G = o(N)$. Then, uniformly in shifts $0 < h \le G$,*

$$C_{f,g}(N,h) = \widehat{f}(1)\widehat{g}(1)N + O_\varepsilon\left(N^\varepsilon\left(DQ + N/G\right)\right).$$

Proof. In fact, the correlations may be expressed as (for all f, g)

$$C_{f,g}(N,h) = \sum_{n \le N} f(n)g(n+h) = \sum_{q \le N+h} g'(q) \sum_{\substack{n \le N \\ n \equiv -h \bmod q}} f(n),$$

in which, if f has range $D \ll N$ and g is G-sifted of range $Q \ll N$, all q are G-sifted, so

$$C_{f,g}(N,h) = \sum_{n \le N} f(n)g(n+h) = \sum_{q \le Q} g'(q)\widehat{f}(1)\frac{N}{q} + O_\varepsilon\left(N^\varepsilon \sum_{q \le Q} |g'(q)| \left(D + \frac{N}{qG}\right)\right),$$

from (22), for the $0 < h \le G$, therefore with $G = o(N)$ we have the equation above. □

Alternative Proof. We might, also, use (23) to estimate our correlation:

$$\begin{aligned} C_{f,g}(N,h) &= \sum_{q \le Q} \widehat{g}(q) \sum_{n \le N} f(n)c_q(n+h) = \sum_{q \le Q} \widehat{g}(q)\mu(q)\widehat{f}(q)N + O_\varepsilon\left(N^\varepsilon D \sum_{q \le Q} |\widehat{g}(q)| q\right), \\ &= \widehat{f}(1)\widehat{g}(1)N + \sum_{G < q \le Q} \widehat{g}(q)\mu(q)\widehat{f}(q)N + O_\varepsilon\left(N^\varepsilon DQ\right) \\ &= \widehat{f}(1)\widehat{g}(1)N + O_\varepsilon\left(N^\varepsilon(DQ + N/G)\right), \end{aligned}$$

where we are still using hypotheses $0 < h \le G$, $G = o(N)$, g is G-sifted of range $Q \ll N$ and f sieve of range $D \ll N$. □

Compared to the results (19) in §6, apart from a short shift (and sifting hypothesis), we have now in the main term a collapse to the lonely $\widehat{f}(1)\widehat{g}(1)$ (the first of $\mathfrak{S}_{f,g}(h)$, since $c_1(h) = 1$).

This is in accordance with heuristics in case $f = g$ above: in fact, our singular series is now, from the hypothesis g is G-sifted,

$$\mathfrak{S}_{f,g}(h) = \widehat{f}(1)\widehat{g}(1) + \sum_{G < \ell \le Q} \widehat{f}(\ell)\widehat{g}(\ell)c_\ell(h) = \widehat{f}(1)\widehat{g}(1) + O_\varepsilon\left(N^\varepsilon/G\right).$$

Anyway, this parameter $G \to \infty$, $G = o(N)$ as $N \to \infty$, in our formulæ, has to be chosen (at least) as $G \gg N^\delta$, for some fixed $\delta > 0$ (compare Remark 7 after Lemma A.5 in §9).

We refer the interested reader to the last section §9, for further properties, involving sieve functions and numbers free of small prime factors, say without prime divisors $p \le G$, especially, to Lemma A.5.

In fact, this Lemma shows that, in sums of sieve functions (with residue $0 < |a| \le G$), the condition of being coprime to a fixed natural number q, which is free of prime factors $p \le G$, may be managed at a small cost (depending, of course, on G).

We leave, for our reader, a simple exercise about this property: with the above considerations starting from (22), apply the third formula of Lemma A.5, §9, in order to prove the noteworthy property, for the, say, *q-coprime correlation* of f and g (for which recover above hypotheses), namely (compare above Proposition 5 for usual correlation)

$$C_{f,g}^{(q)}(N,h) \stackrel{def}{=} \sum_{\substack{n \le N \\ (n,q)=1}} f(n)g(n+h) = \widehat{f}(1)\widehat{g}(1)N + O_{\varepsilon}\left(N^{\varepsilon}(DQ + N/G)\right).$$

This time the error term $N^{1+\varepsilon}/G$ is also the cost we pay to "manage" the condition of coprimality with q, which is free of factors $p \le G$.

We see that, when considering averages of correlations, esp. [3], the really important part of our singular series is the first term, like here with the sifting hypothesis. In some sense, averaging over the shift (compare the symmetry integral calculations in §9) "smooths", so to speak, the arithmetic irregularities, which are "overridden", here, by the "no small primes condition".

8. Concluding remarks

The "shift-Ramanujan expansion", here, is a very special kind of Ramanujan expansion, "entangling" the two single Ramanujan expansions, of f & g (like the definition of f and g correlation entangles them). This is clear in the proof of our Theorem 1, in section 5, and is evident in the formulæ, for the shifted convolution sum of f & g in terms of the (finite-)Ramanujan coefficients $\hat{f}$ & $\hat{g}$: see (20).

In fact, while these formulæ give a duality between f and g entanglement (in their correlation) and $\widehat{f}$ and $\widehat{g}$ entanglement (see formula (20), esp.), we already see a duality in between f and g correlation and f' and g' entanglement, with the introduction of fractional parts, outside main term, in section 2 (see Proposition 1 proof).

By the way, the word "duality", used freely in our context here, has a resemblance in links between different spectral analyses for a problem; in fact, the "spectral formula", for correlation, involved in the heuristic in section 2, is based exactly on the (finite) Fourier expansion for the fractional parts, coming, as usual, from 1-periodic Bernoulli functions (compare [5]). Of course, we also have two kinds of Ramanujan expansions (say, two "harmonic formulæ"): the one for the single f and g, with coefficients $\hat{f}$ and $\hat{g}$, featuring correlations of Ramanujan sums (with two moduli, i.e. (20) in section 6) and, in parallel, the shift-Ramanujan expansion with a kind of "mysterious", new, shift-Ramanujan coefficients times the Ramanujan sum (with one modulus).

These two ways of expanding the correlation of f & g give two possible approaches.

On one hand, the single and finite Ramanujan expansions entangle the two moduli (of $\hat{f}$ & $\hat{g}$) inside the correlation of Ramanujan sums w.r.t. the same two moduli.

On the other hand, the shift-Ramanujan expansion entangles the two functions f & g (so, the moduli of $\hat{f}$ & $\hat{g}$) inside the "black box", given by the shift-Ramanujan coefficients, $\widehat{C_{f,g}}$ (so, at last, shift-Ramanujan expansions have only one modulus, apparently). In some sense, our Theorem 1 tries to take a glance, in suitable hypotheses, to this black box.

The "spectral analysis", we are dealing with, of course, is elementary here, nothing sophisticated like the one for shifted convolution sums, say, in the context of modular forms as in the Rankin–Selberg method. However, we think that there are, almost surely, links to that: compare, for example, our formula for the shifted convolution sums, with a coprimality condition we had at the end of section 7.

In our next paper and in our future investigations, we will try to clarify the notion of regularity for shift-Ramanujan expansions. In particular, we will look for sufficient conditions, giving finiteness of this expansion.

9. Appendix

The results here are listed in increasing order of technicality.

We start, as we use this bound for the next Lemma, with the following elementary property (which we could not locate in the literature). It is the only "Fact", in this paper.

Fact. *For all $a, b \in \mathbb{N}$ we have*

$$\varphi(ab) \le a\varphi(b).$$

Remark 3. A straightforward proof comes considering lattice points of the a times b rectangle.

Proof. We use the well-known

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \quad \forall n \in \mathbb{N},$$

in order to get

$$\frac{\varphi(ab)}{\varphi(b)} = a \prod_{p|ab} \left(1 - \frac{1}{p}\right) \prod_{p|b} \left(1 - \frac{1}{p}\right)^{-1} \le a. \quad \square$$

We prove, as we often use this bound in the above sections, the following Lemma.

Lemma A.1. *For all $q \in \mathbb{N}$ and $n \in \mathbb{Z}$ we have*

$$|c_q(n)| \le (n, q).$$

Proof. We simply put together Hölder's identity [20] $c_q(n) = \varphi(q)\mu(q/(n,q))/\varphi(q/(n,q))$ and the previous fact, with $a = (n,q)$, $b = q/(n,q)$, so the result follows. $\square$

We state and prove a result that we need, but it has an interest of its own.

Lemma A.2. *Let $D, N, t \in \mathbb{N}$ and assume $f : \mathbb{N} \to \mathbb{C}$ is a sieve function of range $D \ll N$. Then, uniformly in $a \in \mathbb{Z}$,*

$$\sum_{\substack{n \le N \\ n \equiv a \bmod t}} f(n) = \frac{N}{t} \sum_{k|t} \widehat{f}(k) c_k(a) + O_\varepsilon\left(D^{1+\varepsilon}\right).$$

Remark 4. We may clearly assume that $0 \le |a| \le t$, whence $a \ll N$.

Proof. Opening $f(n)$, our left hand side is

$$\sum_{d \le D} f'(d) \sum_{\substack{n \le N \\ n \equiv 0 \bmod d \\ n \equiv a \bmod t}} 1 = \sum_{\substack{d \le D \\ (d,t)|a}} f'(d) \left(\frac{N(d,t)}{dt} + O(1) \right),$$

where the remainder is

$$\ll_\varepsilon D^\varepsilon \sum_{\substack{d \le D \\ (d,t)|a}} 1 \ll_\varepsilon D^{1+\varepsilon}.$$

The main term here is

$$\frac{N}{t} \sum_{\substack{d \le D \\ (d,t)|a}} \frac{f'(d)(d,t)}{d}$$

for which we evaluate (using (1) and definitions above)

$$\sum_{\substack{d \le D \\ (d,t)|a}} \frac{f'(d)(d,t)}{d} = \sum_{\substack{\ell|t \\ \ell|a}} \ell \sum_{\substack{d_0 \\ (d_0, t/\ell)=1}} \frac{f'(\ell d_0)}{\ell d_0} = \sum_{\substack{\ell|t \\ \ell|a}} \ell \sum_{m|\frac{t}{\ell}} \mu(m) \sum_{d_1} \frac{f'(\ell m d_1)}{\ell m d_1}$$

$$= \sum_{\substack{\ell|t \\ \ell|a}} \ell \sum_{m|\frac{t}{\ell}} \mu(m) \widehat{f}(\ell m) = \sum_{k|t} \widehat{f}(k) \sum_{\substack{\ell|k \\ \ell|a}} \ell \mu\left(\frac{k}{\ell}\right) = \sum_{k|t} \widehat{f}(k) c_k(a). \quad \square$$

The previous lemma proves the following result, which is also of independent interest.

Lemma A.3. *Let $\ell, D, N \in \mathbb{N}$ and assume $f : \mathbb{N} \to \mathbb{C}$ is a sieve function of range $D \ll N$. Then, uniformly in $a \in \mathbb{Z}$,*

$$\sum_{n \le N} f(n) c_\ell(n-a) = \widehat{f}(\ell) c_\ell(a) N + O_\varepsilon\left((D\ell)^{1+\varepsilon}\right).$$

Proof. Inserting

$$c_\ell(n-a) = \sum_{\substack{t|\ell \\ t|n-a}} t\mu\left(\frac{\ell}{t}\right),$$

(see (1)), the left hand side is (from the previous lemma)

$$\sum_{t|\ell} t\mu\left(\frac{\ell}{t}\right) \sum_{\substack{n\le N \\ n\equiv a \bmod t}} f(n) = N\sum_{t|\ell}\mu\left(\frac{\ell}{t}\right)\sum_{k|t}\widehat{f}(k)c_k(a) + O_\varepsilon\left((D\ell)^{1+\varepsilon}\right),$$

whence we simply get the result, from

$$\sum_{t|\ell}\mu\left(\frac{\ell}{t}\right)\sum_{k|t}\widehat{f}(k)c_k(a) = \sum_{k|\ell}\widehat{f}(k)c_k(a)\sum_{t'|\frac{\ell}{k}}\mu\left(\frac{\ell}{kt'}\right) = \widehat{f}(\ell)c_\ell(a). \quad \square$$

Remark 5. We explicitly point out that it is very important to have any improvement in the remainders of these two lemmas (and it is clear that once A.2 has a better error bound, then A.3 also has). In fact, this is true for both the lemmas and for their applications, especially to the main terms in explicit formulæ, for correlations; in particular, the explicit formula for shift Ramanujan expansion coefficients (compare Theorem 1, section 1) requires a straightforward application (in the proof of Corollary 1, see §5) of Lemma A.3.

We give a very useful lemma, esp., for the shift Ramanujan expansions.

Lemma A.4. *Let $F:\mathbb{N}\to\mathbb{C}$ have a Ramanujan expansion, i.e.*

$$F(h) = \sum_{q=1}^{\infty}\widehat{F}(q)c_q(h),$$

uniformly convergent in $h\in\mathbb{N}$, with certain coefficients $\widehat{F}(q)\in\mathbb{C}$ independent of h (even in their support). Then, these are

$$\widehat{F}(\ell) = \frac{1}{\varphi(\ell)}\lim_{x\to\infty}\frac{1}{x}\sum_{h\le x}F(h)c_\ell(h). \tag{24}$$

Remark 6. We call the above "Carmichael's formula", for Ramanujan coefficients [1], [20].

Proof. Fix $\ell\in\mathbb{N}$ and, by uniform convergence, we have $\forall\varepsilon>0\ \exists Q=Q(\varepsilon,\ell)$, with $Q>\ell$ and

$$\left|\sum_{q>Q}\widehat{F}(q)c_q(h)\right| < \frac{\varepsilon}{d(\ell)},$$

entailing

$$\frac{1}{x}\sum_{h\le x}F(h)c_\ell(h) = \sum_{q\le Q}\widehat{F}(q)\frac{1}{x}\sum_{h\le x}c_\ell(h)c_q(h) + \frac{1}{x}\sum_{h\le x}c_\ell(h)\sum_{q>Q}\widehat{F}(q)c_q(h)$$

(notice purity allows the exchange of sums) implies ("$\lim_x$", here, abbreviating "$\lim_{x\to\infty}$")

$$\left|\frac{1}{\varphi(\ell)}\lim_x\frac{1}{x}\sum_{h\le x}F(h)c_\ell(h) - \frac{1}{\varphi(\ell)}\sum_{q\le Q}\widehat{F}(q)\lim_x\frac{1}{x}\sum_{h\le x}c_\ell(h)c_q(h)\right|$$
$$\le \frac{\varepsilon}{\varphi(\ell)d(\ell)}\lim_x\frac{1}{x}\sum_{h\le x}(\ell,h),$$

from $|c_\ell(h)|\le(\ell,h)$, see Lemma A.1, whence the orthogonality relations (see [20])

$$\lim_x\frac{1}{x}\sum_{h\le x}c_\ell(h)c_q(h) = \sum_{j\in\mathbb{Z}_\ell^*}\sum_{r\in\mathbb{Z}_q^*}\lim_x\frac{1}{x}\sum_{h\le x}e^{2\pi i(j/\ell-r/q)h} = \sum_{j\in\mathbb{Z}_\ell^*}\sum_{r\in\mathbb{Z}_q^*}\mathbf{1}_{q=\ell}\mathbf{1}_{r=j} = \mathbf{1}_{q=\ell}\varphi(\ell)$$

and the formula

$$\frac{1}{x}\sum_{h\le x}(\ell,h) = \sum_{t|\ell}\frac{t}{x}\sum_{\substack{h'\le\frac{x}{t}\\(h',\frac{\ell}{t})=1}}1 = \sum_{t|\ell}\frac{t}{x}\sum_{d|\frac{\ell}{t}}\mu(d)\left[\frac{x}{dt}\right] = \sum_{t|\ell}\sum_{d|\frac{\ell}{t}}\frac{\mu(d)}{d} + O\left(\frac{1}{x}\sum_{t|\ell}td(\ell/t)\right)$$
$$= \sum_{t|\ell}\frac{\varphi(\ell/t)}{\ell/t} + o(1) = \sum_{d|\ell}\frac{\varphi(d)}{d} + o(1)$$

(flipping to the complementary divisor $d=\ell/t$), as $x\to\infty$, together give

$$\left|\frac{1}{\varphi(\ell)}\lim_{x\to\infty}\frac{1}{x}\sum_{h\le x}F(h)c_\ell(h) - \widehat{F}(\ell)\right| \le \frac{\varepsilon}{\varphi(\ell)d(\ell)}\sum_{d|\ell}\frac{\varphi(d)}{d} \le \frac{\varepsilon}{\varphi(\ell)} \le \varepsilon,$$

which, as $\varepsilon>0$ is arbitrary, shows (24). $\square$

We give, now, an important result, with applications to the G-sifted functions.

Lemma A.5. *Let $G, q, N\in\mathbb{N}$ and assume $f:\mathbb{N}\to\mathbb{C}$ is essentially bounded, while q has all prime factors greater than G, i.e., $(q,P(G))=1$. Then*

$$\sum_{\substack{n\le N\\(n,q)=1}} f(n) = \sum_{n\le N} f(n) + O_\varepsilon\left((qN)^\varepsilon \frac{N}{G}\right).$$

Furthermore, assuming $0<|a|\le G$,

$$\sum_{\substack{n\le N\\ n\equiv a \bmod t\\(n,q)=1}} f(n) = \sum_{\substack{n\le N\\ n\equiv a \bmod t}} f(n) + O_\varepsilon\left((qN)^\varepsilon\left(\frac{N}{tG}+1\right)\right),$$

whence, adding the hypothesis f is a sieve function of range $D\ll N$, from Lemma A.2,

$$\sum_{\substack{n\le N\\ n\equiv a \bmod t\\(n,q)=1}} f(n) = \frac{N}{t}\sum_{k|t}\widehat{f}(k)c_k(a) + O_\varepsilon\left((qN)^\varepsilon\left(\frac{N}{tG}+D\right)\right);$$

also, sieve f of range $D\ll N$ have, in case $0<|a|\le G$,

$$\sum_{\substack{n\le N\\(n,q)=1}} f(n)c_\ell(n-a) = \widehat{f}(\ell)c_\ell(a)N + O_\varepsilon\left((\ell qN)^\varepsilon\left(\frac{N}{G}+D\ell\right)\right)$$

whence, applying our Lemma A.3 for case $0<|a|\le G$,

$$\sum_{\substack{n\le N\\(n,q)=1}} f(n)c_\ell(n-a) = \sum_{n\le N} f(n)c_\ell(n-a) + O_\varepsilon\left((\ell qN)^\varepsilon\left(\frac{N}{G}+D\ell\right)\right).$$

Proof. Lemma 3 of [8], $\mathbf{1}_{(n,q)=1}=\sum_{d|q,d|n}\mu(d)$, entails first LHS is (after, use: $d(q)\ll_\varepsilon q^\varepsilon$)

$$\sum_{d|q}\mu(d)\sum_{m\le \frac{N}{d}} f(dm) = \sum_{n\le N} f(n) + O_\varepsilon\Big(N^\varepsilon \sum_{\substack{d|q\\d>G}}\frac{N}{d}\Big) = \sum_{n\le N} f(n) + O_\varepsilon\Big(N^{1+\varepsilon}\frac{d(q)}{G}\Big).$$

In the same way

$$\sum_{\substack{n\le N\\ n\equiv a \bmod t\\(n,q)=1}} f(n) = \sum_{d|q}\mu(d)\sum_{\substack{n\le N\\ n\equiv a \bmod t\\ n\equiv 0 \bmod d}} f(n) = \sum_{\substack{n\le N\\ n\equiv a \bmod t}} f(n) + O_\varepsilon\Big(N^\varepsilon\sum_{\substack{d|q\\d>G\\(d,t)|a}}\left(\frac{N(d,t)}{dt}+1\right)\Big),$$

but, now, $(d,t)=1$, since $(d,t)>1$ implies the primes $p|(d,t)$ are, dividing $a\neq 0$, all not greater than G, which is absurd by our assumption. Thus we have proved our second formula. In turn, this can be used to prove our fourth formula (compare Lemma A.3 proof),

$$\sum_{\substack{n\le N\\(n,q)=1}} f(n)c_\ell(n-a) = \sum_{t|\ell} t\mu\left(\frac{\ell}{t}\right) \sum_{\substack{n\le N\\ n\equiv a \bmod t}} f(n) + O_\varepsilon\left((\ell q N)^\varepsilon (N/G + D\ell)\right)$$
$$= \widehat{f}(\ell)c_\ell(a)N + O_\varepsilon\left((\ell q N)^\varepsilon (N/G + D\ell)\right). \quad \square$$

Remark 7. Since $\varepsilon > 0$ is arbitrarily small, $G \gg N^\delta$ ($\delta > 0$ fixed) is preferred in all formulæ.

We express, here, the singular series in a very useful way, for heuristic calculations above.

Lemma A.6. *Let $h \ge 0$ be integer and assume $f, g : \mathbb{N} \to \mathbb{C}$ have finite Ramanujan expansions. Then*

$$\mathfrak{S}_{f,g}(h) = \sum_{l|h} l \sum_d \frac{f'(d)}{d} \sum_{\substack{q\\(q,d)=l}} \frac{g'(q)}{q}.$$

Proof. In fact, by the definition of $\mathfrak{S}_{f,g}$, as well as equations (1) and (10) give (when $h = 0$, $l|h$ means any $l \ge 1$):

$$\mathfrak{S}_{f,g}(h) = \sum_{q=1}^{\infty} \widehat{f}(q)\widehat{g}(q)c_q(h) = \sum_{q=1}^{\infty} \widehat{f}(q)\widehat{g}(q) \sum_{\substack{l|h\\ l|q}} l\mu\left(\frac{q}{l}\right) = \sum_{l|h} l \sum_k \mu(k)\widehat{f}(lk)\widehat{g}(lk)$$
$$= \sum_{l|h} l \sum_k \mu(k) \sum_d \frac{f'(lkd)}{lkd} \sum_q \frac{g'(lkq)}{lkq} = \sum_{l|h} \sum\sum_{(t,r)=1} \frac{f'(lt)g'(lr)}{ltr}$$
$$= \sum_{l|h} l \sum\sum_{(d,q)=l} \frac{f'(d)g'(q)}{dq}. \quad \square$$

We explore, now, the properties of a particular arithmetic function, so as to clarify the properties of regularity (see soon after Theorem 1, in §1) for the shift-Ramanujan expansions.

We build here an example of an arithmetic function, for which we prove that its s.R.e. is not in first class.[4] It is denoted f_H, as it depends on the "length", of short intervals $[x - H, x + H]$ for which we check the symmetry (in particular, for almost all of them, namely all $x \in (N, 2N]$, save at most $o(N)$ of them, as $N \to \infty$); we may define it, generically, as assuming only two different values $c_1, c_2 \in \mathbb{C}$, in consecutive intervals with length H and it is periodic of period $2H$ (it clearly has mean-value $(c_1 + c_2)/2$ and is not constant, as $c_1 \ne c_2$).

However, its "short interval mean-value"has sometimes large size exactly when the short interval length is H; so, we don't consider its "Selberg integral":

[4] This, in turn, proves it's not regular, since (iii) of Theorem 1 $\Rightarrow$ s.R.e. is in δ-class $\forall\delta \in (0, 1]$.

$$J_{f_H}(N,H) \stackrel{def}{=} \sum_{N<x\le 2N} \Big| \sum_{x<n\le x+H} f_H(n) - M_{f_H}(x,H) \Big|^2,$$

as the difficulties show up already for the definition of this "expected mean-value", in short intervals, $M_{f_H}(x,H)$. (Compare [3], [6], for the definitions and following properties.)

Thus we consider the "easier" symmetry integral ($\mathrm{sgn}(0) \stackrel{def}{=} 0$, $r \neq 0 \Rightarrow \mathrm{sgn}(r) \stackrel{def}{=} \frac{|r|}{r}$)

$$J_{f_H\,,\,\mathrm{sgn}}(N,H) \stackrel{def}{=} \sum_{N<x\le 2N} \Big| \sum_{x-H\le n\le x+H} \mathrm{sgn}(n-x) f_H(n) \Big|^2,$$

that checks the symmetry, around x, in *almost all* the short intervals $[x-H, x+H]$; namely, neglecting at most $o(N)$ of them for $x \in (N, 2N]$ and the length is *short*, i.e. $H = o(N)$, as $N \to \infty$. The short intervals expected mean-value vanishes, here in the f symmetry integral, whatever the $f : \mathbb{N} \to \mathbb{C}$ is (and this by definition).

We know (from Lemma 7 of [3]) that, since f_H is bounded, hereafter $H = o(N)$,

$$J_{f_H\,,\,\mathrm{sgn}}(N,H) = \sum_h W_H(h) \sum_{N<n\le 2N} f_H(n) f_H(n-h) + O(H^3),$$

where the weight $W_H(h)$ is the correlation of sgn function, [6], namely

$$W_H(h) \stackrel{def}{=} \sum_{\substack{-H\le h_1\,,\,h_2\le H \\ h_2-h_1=h}} \sum \mathrm{sgn}(h_1)\mathrm{sgn}(h_2),$$

which we will not write explicitly (see [6]), but only use its properties [6], like: W_H is even,

$$W_H(h) \ll H, \quad W_H(h) = 0, \forall h \notin [-2H, 2H].$$

We have a difference with our correlations, namely for $h < 0$, abbreviating C_{f_H} for C_{f_H,f_H},

$$\sum_{N<n\le 2N} f_H(n) f_H(n-h) = C_{f_H}(2N,-h) - C_{f_H}(N,-h),$$

while the case $h > 0$ (and $h \le 2H$, here) is handled from the immediate change of variables

$$\begin{aligned} \sum_{N<n\le 2N} f_H(n) f_H(n-h) &= \sum_{N-h<n\le 2N-h} f_H(n) f_H(n+h) \\ &= C_{f_H}(2N,h) - C_{f_H}(N,h) + O(H) \end{aligned}$$

and the case $h = 0$ is negligible:

$$\sum_{N<n\le 2N} f_H^2(n) \ll N,$$

giving in all a new formula involving our correlations (from previous sections). Thus,

$$J_{f_H\,,\,\mathrm{sgn}}(N,H) = 2\sum_{h>0} W_H(h)\left(C_{f_H}(2N,h) - C_{f_H}(N,h)\right) + O(NH+H^3). \tag{25}$$

We leave, as an exercise, to prove from the definition

$$J_{f_H\,,\,\mathrm{sgn}}(N,H) \gg NH^2,$$

say, the f_H symmetry integral is "trivial". However, assuming f_H is in the first class,

$$C_{f_H}(N,h) = \sum_{\ell\ll N} \widehat{C_{f_H}}(N,\ell)c_\ell(h), \quad \text{with} \quad \widehat{C_{f_H}}(N,\ell) \ll_\varepsilon \frac{N^{1+\varepsilon}}{\ell^2},$$

$$C_{f_H}(2N,h) = \sum_{\ell\ll N} \widehat{C_{f_H}}(2N,\ell)c_\ell(h), \quad \text{with} \quad \widehat{C_{f_H}}(2N,\ell) \ll_\varepsilon \frac{N^{1+\varepsilon}}{\ell^2},$$

entailing

$$2\sum_{h>0} W_H(h)\left(C_{f_H}(2N,h) - C_{f_H}(N,h)\right) \ll_\varepsilon \sum_{\ell\ll N} \frac{N^{1+\varepsilon}}{\ell^2}\left|\sum_{h>0} W_H(h)c_\ell(h)\right|,$$

in which, defining as usual $e_q(n) \overset{def}{=} e^{2\pi i n/q}$ (the usual *additive characters* modulo q)

$$\sum_{h>0} W_H(h)c_\ell(h) = \frac{1}{2}\sum_h W_H(h)c_\ell(h) - H\varphi(\ell) = \frac{1}{2}\sum_{j\in\mathbb{Z}_\ell^*}\sum_h W_H(h)e_\ell(jh) - H\varphi(\ell)$$

using the $W_H(h)$ positive exponential sums [6] (with the additive characters orthogonality),

$$\sum_{h>0} W_H(h)c_\ell(h) \ll \sum_{j\le\ell}\sum_h W_H(h)e_\ell(jh) + H\varphi(\ell) \ll \ell \sum_{h\equiv 0 \bmod \ell} W_H(h) + H\varphi(\ell) \ll \ell H$$

(uniformly in $\ell > 1$, otherwise $c_1(h) = 1$ gives $\sum_h W_H(h) = 0$, compare [6]), so from (25) we get

$$J_{f_H\,,\,\mathrm{sgn}}(N,H) \ll_\varepsilon N^{1+\varepsilon}H + H^3,$$

say, in case $H \ll \sqrt{N}$,

$$NH^2 \ll J_{f_H\,,\,\mathrm{sgn}}(N,H) \ll_\varepsilon N^{1+\varepsilon}H,$$

a contradiction whenever $N^{2\varepsilon} \ll H \ll \sqrt{N}$.

Thus f_H we built above is not in first class. Its shift Ramanujan expansion is not regular and we guess that the real problem, here, is the dependence of our arithmetic function on the "external" parameter H.

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