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AN ASYMPTOTIC FORMULA FOR THE COEFFICIENTS OF $j(z)$

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We obtain a new proof of an asymptotic formula for the coefficients of the j-invariant of elliptic curves. Our proof does not use the circle method. We use Laplace's method of steepest descent and the Hardy–Ramanujan asymptotic formula for the partition function. (The latter asymptotic formula can be derived without the circle method.)

Keywords: Asymptotic formula; Fourier coefficients; j-invariant; weakly holomorphic modular forms.

Mathematics Subject Classification 2010: 11F30

1. Introduction

Suppose k is an even integer and let M_k denote the space of weight k holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$. It is well known that the algebra of modular forms on $\mathrm{SL}_2(\mathbb{Z})$ is generated by Eisenstein series of the form

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$ and $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$ and $q = e^{2\pi i z}$. The first non-trivial cusp form is

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}.$$

With this notation, $\Delta \in M_{12}$ and $E_k \in M_k$ whenever $k \geq 4$ is even.

The modular function

$$j(z) = \frac{E_{12}(z)}{\Delta(z)} + \frac{432000}{691} = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n) q^n$$

is a fundamental object in number theory. Petersson [7], and later Rademacher [8] independently, used the circle method to prove the asymptotic formula

$$c(n) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}\, n^{3/4}}, \tag{1.1}$$

as $n \to \infty$.

Petersson and Rademacher were inspired by the seminal work of Hardy and Ramanujan [4] which introduced the circle method in order to prove the asymptotic formula for the partition function

$$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}\, n}, \tag{1.2}$$

as $n \to \infty$, where $p(n)$ is determined by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} (1-q^n)^{-1}.$$

In a recent article [2] we gave a new proof of (1.2) without using the circle method. Our derivation of (1.2) used an algebraic formula of Bruinier and Ono [1].

The purpose of the present article is to give a new proof of (1.1) without using the circle method. We use Laplace's method of steepest descent and the Hardy–Ramanujan asymptotic. In fact, our method yields much more general conclusions.

Let $M_k^!$ denote the space of weight k *weakly holomorphic* modular forms on $\mathrm{SL}_2(\mathbb{Z})$. That is, meromorphic modular forms whose only poles (if any) are at $i\infty$. If $f \in M_k^!$ has $\mathrm{ord}_{i\infty} f = -m < 0$, then

$$f = \sum_{j=0}^{\lfloor k/12 \rfloor + m} b_j E_{k+12(m-j)} \Delta^{j-m}, \tag{1.3}$$

for some $b_j \in \mathbb{C}$ where $b_0 \neq 0$. We have the following theorem which immediately implies (1.1).

Theorem 1. *Suppose $k \in 2\mathbb{Z}$ and $f \in M_k^!$ has $\mathrm{ord}_{i\infty} f = -m < 0$ and $f = \sum_{n=-m}^{\infty} \lambda_f(n)q^n$. Then*

$$\lambda_f(n) \sim i^k \frac{\lambda_f(-m)}{\sqrt{2n}} \left(\frac{n}{m}\right)^{\frac{k}{2}-\frac{1}{4}} e^{4\pi\sqrt{nm}}.$$

Our proof of Theorem 1 has two main steps. The first — which may be of independent interest — uses Laplace's method to prove the following.

Theorem 2. *Suppose*

$$f(z) = \sum_{n=0}^{\infty} \lambda_f(n)q^n,$$

$$g(z) = \sum_{n=0}^{\infty} \lambda_g(n)q^n,$$

where

$$\begin{aligned} \lambda_f(n) &\sim c_f n^{\alpha} e^{A\sqrt{n}}, \\ \lambda_g(n) &\sim c_g n^{\beta} e^{B\sqrt{n}}, \end{aligned} \tag{1.4}$$

with α, β, A, B, c_f, $c_g \in \mathbb{R}$ *and* A, B, c_f, $c_g > 0$. *Then for* $fg(z) = \sum_{n=0}^{\infty} \lambda_{fg}(n) q^n$ *we have*

$$\lambda_{fg}(n) \sim c_f c_g 2\sqrt{2\pi} \frac{A^{2\alpha+1} B^{2\beta+1}}{(A^2+B^2)^{\frac{5}{4}+\alpha+\beta}} n^{\alpha+\beta+\frac{3}{4}} e^{\sqrt{A^2+B^2}\sqrt{n}}.$$

Here is the strategy to prove Theorem 1. Set

$$\sum_{n=0}^{\infty} p^{(j)}(n) q^n = \prod_{n=0}^{\infty} (1-q^n)^{-j}.$$

By Theorem 2 and (1.2), an easy induction shows that $p^{(j)}(n) \sim c_j n^{\alpha_j} e^{A_j\sqrt{n}}$ where

$$\begin{aligned} c_j &= \frac{1}{\sqrt{2}} \left(\frac{j}{24}\right)^{\frac{j+1}{4}}, \\ \alpha_j &= -\frac{j}{4} - \frac{3}{4}, \\ A_j &= \pi\sqrt{\frac{2j}{3}}. \end{aligned}$$

Thus for any fixed integer $m > 0$, the coefficients of

$$q^m \Delta^{-m} = \prod_{n=1}^{\infty} (1-q^n)^{-24m} = \sum_{n=0}^{\infty} p^{(24m)}(n) q^n$$

satisfy the asymptotic formula

$$p^{(24m)}(n) \sim \frac{1}{\sqrt{2n}} \left(\frac{m}{n}\right)^{6m+\frac{1}{4}} e^{4\pi\sqrt{nm}}. \tag{1.5}$$

In light of (1.3), we prove the following.

Theorem 3. *Suppose $k \geq 4$ is even and*

$$f(z) = \sum_{n=0}^{\infty} \lambda_f(n) q^n,$$

where $\lambda_f(n) \geq 0$ for all n and

$$\lambda_f(n) \sim \frac{c_f e^{A\sqrt{n}}}{n^{\alpha}}, \tag{1.6}$$

with real numbers c_f, A, $\alpha > 0$. Then for $fE_k(z) = \sum_{n=0}^{\infty} \lambda_{fE_k}(n) q^n$ we have

$$\lambda_{fE_k}(n) \sim \frac{c_f e^{A\sqrt{n}}}{n^{\alpha-\frac{k}{2}}} \left(\frac{4\pi i}{A}\right)^k.$$

The proof of Theorem 1 follows easily from Theorems 2 and 3.

Proof of Theorem 1. Let $\lambda_{E_{k+12(m-j)}\Delta^{j-m}}(n)$ denote the nth coefficient of $E_{k+12(m-j)}\Delta^{j-m}$. For fixed m and j, we obviously have

$$\lambda_{E_{k+12(m-j)}\Delta^{j-m}}(n) \sim \lambda_{E_{k+12(m-j)}\Delta^{j-m}}(n+j-m) = \lambda_{E_{k+12(m-j)}q^{m-j}\Delta^{j-m}}(n).$$

Hence by (1.3), Theorem 3 and (1.5), we have

$$\begin{aligned}\lambda_f(n) &= \sum_{j=0}^{\lfloor k/12\rfloor+m} b_j \lambda_{E_{k+12(m-j)}\Delta^{j-m}}(n)\\ &\sim \sum_{j=0}^{m-1} b_j \frac{i^k}{\sqrt{2n}}\left(\frac{n}{m-j}\right)^{\frac{k}{2}-\frac{1}{4}} e^{4\pi\sqrt{n(m-j)}} + O(n^{k-1})\\ &\sim b_0 \frac{i^k}{\sqrt{2n}}\left(\frac{n}{m}\right)^{\frac{k}{2}-\frac{1}{4}} e^{4\pi\sqrt{nm}},\end{aligned}$$

since the $e^{4\pi\sqrt{nm}}$ term dominates all of the other exponentials. Finally, observe that $b_0 = \lambda_f(-m)$. □

2. Proof of Theorem 2

The key to proving Theorem 2 is that $\lambda_{fg}(n) = \sum_j \lambda_f(j)\lambda_g(n-j)$ is approximated by

$$c_f c_g \sum_j j^\alpha (n-j)^\beta e^{A\sqrt{j}+B\sqrt{n-j}} = c_f c_g n^{\alpha+\beta} \sum_j G\left(\frac{j}{n}\right) e^{\sqrt{n}F\left(\frac{j}{n}\right)},$$

where

$$\begin{aligned}F(x) &:= A\sqrt{x} + B\sqrt{1-x} : (0,1) \to \mathbb{R}_{>0},\\ G(x) &:= x^\alpha(1-x)^\beta : (0,1) \to \mathbb{R}_{>0}.\end{aligned}$$

Set

$$c := \frac{A^2}{A^2+B^2}.$$

The function $F(x)$ is increasing on $(0,c)$, has a maximum of $\sqrt{A^2+B^2}$ at $x = c$, and is decreasing on $(c,1)$.

Proof of Theorem 2. Let $0 < \epsilon < 1$ be given. By continuity, there exists $\delta > 0$ such that if $|x-c| < 2\delta$, then

$$(1-\epsilon)G(c) < G(x) < (1+\epsilon)G(c). \tag{2.1}$$

We may assume that both $\delta < c$ and $\delta < 1-c$. By (1.4), for large enough n,

$$\begin{aligned}(1-\epsilon)c_f n^\alpha e^{A\sqrt{n}} &< \lambda_f(n) < (1+\epsilon)c_f n^\alpha e^{A\sqrt{n}},\\ (1-\epsilon)c_g n^\beta e^{B\sqrt{n}} &< \lambda_g(n) < (1+\epsilon)c_g n^\beta e^{B\sqrt{n}}.\end{aligned} \tag{2.2}$$

Decompose

$$\begin{aligned}\lambda_{fg}(n) &= \underbrace{\sum_{j=0}^{\lceil (c-\delta)n\rceil -1} \lambda_f(j)\lambda_g(n-j)}_{=:\, S_0(n)} + \underbrace{\sum_{j=\lceil (c-\delta)n\rceil}^{\lfloor (c+\delta)n\rfloor} \lambda_f(j)\lambda_g(n-j)}_{=:\, S_1(n)} \\ &\quad + \underbrace{\sum_{j=\lfloor (c+\delta)n\rfloor+1}^{n} \lambda_f(j)\lambda_g(n-j)}_{=:\, S_2(n)}.\end{aligned}$$

By (1.4),

$$\begin{aligned}S_0(n) &= O\left(n^{|\alpha|+|\beta|}\sum_{j=0}^{\lceil (c-\delta)n\rceil -1} e^{\sqrt{n}F\left(\frac{j}{n}\right)}\right) = O(n^{|\alpha|+|\beta|+1}e^{\sqrt{n}F(c-\delta)}) \\ &= o(n^{\alpha+\beta+3/4}e^{\sqrt{n}F(c)}).\end{aligned}$$

Similarly,

$$S_2(n) = O(n^{|\alpha|+|\beta|+1}e^{\sqrt{n}F(c+\delta)}) = o(n^{\alpha+\beta+3/4}e^{\sqrt{n}F(c)}).$$

It remains to consider $S_1(n)$. For large enough n, the inequalities (2.2) apply to every summand in $S_1(n)$:

$$\begin{aligned}&(1-\epsilon)^2 c_f c_g n^{\alpha+\beta} \sum_{j=\lceil (c-\delta)n\rceil}^{\lfloor (c+\delta)n\rfloor} G\left(\frac{j}{n}\right) e^{\sqrt{n}F\left(\frac{j}{n}\right)} \\ &\quad < S_1(n) < (1+\epsilon)^2 c_f c_g n^{\alpha+\beta} \sum_{j=\lceil (c-\delta)n\rceil}^{\lfloor (c+\delta)n\rfloor} G\left(\frac{j}{n}\right) e^{\sqrt{n}F\left(\frac{j}{n}\right)}.\end{aligned}$$

By (2.1), we have

$$\begin{aligned}&(1-\epsilon)^3 c_f c_g G(c) n^{\alpha+\beta+1} \sum_{j=\lceil (c-\delta)n\rceil}^{\lfloor (c+\delta)n\rfloor} e^{\sqrt{n}F\left(\frac{j}{n}\right)} \cdot \frac{1}{n} \\ &\quad < S_1(n) < (1+\epsilon)^3 c_f c_g G(c) n^{\alpha+\beta+1} \sum_{j=\lceil (c-\delta)n\rceil}^{\lfloor (c+\delta)n\rfloor} e^{\sqrt{n}F\left(\frac{j}{n}\right)} \cdot \frac{1}{n}. \qquad (2.3)\end{aligned}$$

We now consider

$$\widehat{S}_1(n) := \sum_{j=\lceil (c-\delta)n\rceil}^{\lfloor (c+\delta)n\rfloor} e^{\sqrt{n}F\left(\frac{j}{n}\right)} \cdot \frac{1}{n}$$

and compare it to the integral

$$I(n) := \int_{c-\delta}^{c+\delta} e^{\sqrt{n}F(x)} dx.$$

Lemma 4. *Let* $\widehat{S}_1(n)$ *and* $I(n)$ *be as above. Then*

$$I(n) - \frac{e^{\sqrt{n}F(c)}}{n} \le \widehat{S}_1(n) \le I(n) + \frac{e^{\sqrt{n}F(c)}}{n}.$$

Proof. First observe that

$$I(n) \le \int_{\frac{\lceil (c-\delta)n\rceil}{n} - \frac{1}{n}}^{\frac{\lfloor (c+\delta)n\rfloor}{n} + \frac{1}{n}} e^{\sqrt{n}F(x)}dx =: I_{\text{over}}(n).$$

Recall that right-end-point Riemann sums overestimate integrals of increasing functions and left-end-point Riemann sums overestimate integrals of decreasing functions. Thus we obtain an upper bound for $I_{\text{over}}(n)$ by constructing a Riemann sum (of rectangles of width $1/n$) with right end-points for the interval to the left of c and with left end-points for the interval to the right of c. In particular

$$I(n) \le I_{\text{over}}(n) \le \widehat{S}_1(n) + \frac{e^{\sqrt{n}F(c)}}{n},$$

where the $\frac{e^{\sqrt{n}F(c)}}{n}$ term must be added to cover the gap between the right-end-point rectangles and the left-end-point rectangles.

Similarly, we observe that

$$I(n) \ge \int_{\frac{\lceil (c-\delta)n\rceil}{n}}^{\frac{\lfloor (c+\delta)n\rfloor}{n}} e^{\sqrt{n}F(x)}dx =: I_{\text{under}}(n).$$

An underestimate for I_{under} is obtained by constructing a Riemann sum with left end-points for the interval to the left of c and right end-points to the right of c. In particular,

$$I(n) \ge I_{\text{under}}(n) \ge \widehat{S}_1(n) - \frac{e^{\sqrt{n}F(c)}}{n},$$

where the $-\frac{e^{\sqrt{n}F(c)}}{n}$ term compensates for the overlap between the left-end-point rectangles and the right-end-point rectangles. The lemma follows immediately. □

Laplace's method (see, for example, [3, Chap. 19.3]) implies that

$$I(n) \sim \sqrt{\frac{2\pi}{\sqrt{n}|F''(c)|}} \cdot e^{\sqrt{n}F(c)}.$$

That is, for large enough n we have

$$(1-\epsilon)\sqrt{\frac{2\pi}{|F''(c)|}}n^{-1/4}e^{\sqrt{n}F(c)} < I(n) < (1+\epsilon)\sqrt{\frac{2\pi}{|F''(c)|}}n^{-1/4}e^{\sqrt{n}F(c)}. \tag{2.4}$$

Combining this with (2.3) and Lemma 4 gives

$$\begin{aligned} S_1(n) &< (1+\epsilon)^3 c_f c_g G(c) n^{\alpha+\beta+1}\left\{ I(n) + \frac{e^{\sqrt{n}F(c)}}{n}\right\} \\ &< (1+\epsilon)^3 c_f c_g G(c) n^{\alpha+\beta+1}\left\{ (1+\epsilon)\sqrt{\frac{2\pi}{|F''(c)|}} n^{-1/4} e^{\sqrt{n}F(c)} + \frac{e^{\sqrt{n}F(c)}}{n}\right\} \\ &= (1+\epsilon)^4 c_f c_g G(c) \sqrt{\frac{2\pi}{|F''(c)|}} n^{\alpha+\beta+\frac{3}{4}} e^{\sqrt{n}F(c)} + o(n^{\alpha+\beta+\frac{3}{4}} e^{\sqrt{n}F(c)}) \end{aligned}$$

and similarly

$$S_1(n) > (1-\epsilon)^4 c_f c_g G(c) \sqrt{\frac{2\pi}{|F''(c)|}} n^{\alpha+\beta+\frac{3}{4}} e^{\sqrt{n}F(c)} + o(n^{\alpha+\beta+\frac{3}{4}} e^{\sqrt{n}F(c)}).$$

We conclude that

$$S(n) \sim S_1(n) \sim c_f c_g G(c) \sqrt{\frac{2\pi}{|F''(c)|}} n^{\alpha+\beta+\frac{3}{4}} e^{\sqrt{n}F(c)}.$$

Finally, it is elementary that $F(c) = \sqrt{A^2+B^2}$, that $|F''(c)| = \frac{(A^2+B^2)^{5/2}}{4A^2B^2}$, and that $G(c) = \frac{A^{2\alpha}B^{2\beta}}{(A^2+B^2)^{\alpha+\beta}}$.

3. Proof of Theorem 3

It is convenient to set $g = i^k(E_k - 1) = \sum_{n=1}^{\infty} \lambda_g(n) q^n$ so that $\lambda_g(n) \geq 0$. We will show that for $fg = \sum_{n=0}^{\infty} \lambda_{fg}(n) q^n$ we have

$$\lambda_{fg}(n) \sim \frac{c_f e^{A\sqrt{n}}}{n^{\alpha-\frac{k}{2}}}\left(\frac{4\pi}{A}\right)^k. \tag{3.1}$$

Once (3.1) is established, it is easy to see that $fE_k = f + i^k fg$ and so

$$\begin{aligned} \lambda_{fE_k}(n) &\sim \lambda_f(n) + i^k \lambda_{fg}(n) \\ &\sim \frac{c_f e^{A\sqrt{n}}}{n^{\alpha}} + i^k \frac{c_f e^{A\sqrt{n}}}{n^{\alpha-\frac{k}{2}}}\left(\frac{4\pi}{A}\right)^k \\ &\sim \frac{c_f e^{A\sqrt{n}}}{n^{\alpha-\frac{k}{2}}}\left(\frac{4\pi i}{A}\right)^k, \end{aligned}$$

as desired. We now prove (3.1).

Once again, the key observation is that $\lambda_{fg}(n) = \sum_j \lambda_f(n-j)\lambda_g(j)$ is approximated by

$$\frac{c_f}{n^\alpha} \sum_j G\left(\frac{j}{n}\right) e^{A\sqrt{n}F(\frac{j}{n})} \lambda_g(j),$$

where

$$F(x) := \sqrt{1-x} : (0,1) \to \mathbb{R}_{>0},$$
$$G(x) := \frac{1}{(1-x)^\alpha} : (0,1) \to \mathbb{R}_{>0}.$$

Elementary calculus (Taylor's theorem) gives that for $0 \le x \le \delta < 1$,

$$1 - \frac{x}{2} - \frac{\delta^2}{8(1-\delta)^{3/2}} \le F(x) \le 1 - \frac{x}{2}, \tag{3.2}$$

$$1 \le G(x) \le 1 + \alpha\frac{\delta}{(1-\delta)^{\alpha+1}}. \tag{3.3}$$

Since $e^{A\sqrt{n}x}$ is an increasing function of x, (3.2) implies

$$e^{A\sqrt{n}\left\{1-\frac{x}{2}-\frac{\delta^2}{8(1-\delta)^{3/2}}\right\}} \le e^{A\sqrt{n}F(x)} \le e^{A\sqrt{n}\left\{1-\frac{x}{2}\right\}}$$

for all $0 \le x \le \delta < 1$. In particular, for integers $0 \le j \le \delta n$, take $x = j/n$ above to deduce

$$e^{A\sqrt{n}-\frac{Aj}{2\sqrt{n}}-\frac{A\sqrt{n}\delta^2}{8(1-\delta)^{3/2}}} \le e^{A\sqrt{n-j}} \le e^{A\sqrt{n}-\frac{Aj}{2\sqrt{n}}}. \tag{3.4}$$

We will establish (3.1) in two steps. We first show that

$$\limsup_{n\to\infty} \frac{\lambda_{fg}(n)}{c_f\left(\frac{4\pi}{A}\right)^k \frac{e^{A\sqrt{n}}}{n^{\alpha-k/2}}} \le 1. \tag{3.5}$$

Let $\epsilon > 0$ be given. By continuity, we can fix $0 < \delta < 1$ such that for $0 \le x \le \delta$ we have

$$G(x) \le (1+\epsilon). \tag{3.6}$$

By assumption, $\lambda_f(n) = O(e^{A\sqrt{n}})$ and $\lambda_g(n) = O(n^{k-1})$. Hence $\lambda_f(n-j)\lambda_g(j) = O(e^{A\sqrt{n-j}}j^{k-1})$ and so

$$\sum_{j=\lfloor\delta n\rfloor+1}^{n} \lambda_f(n-j)\lambda_g(j) = O(e^{A\sqrt{n-n\delta}}n^k) = O(e^{A\sqrt{1-\delta}\sqrt{n}}n^k) = o\left(\frac{e^{A\sqrt{n}}}{n^{\alpha-k/2}}\right).$$

Thus, it suffices to consider

$$S_\delta(n) := \sum_{j=0}^{\lfloor\delta n\rfloor} \lambda_f(n-j)\lambda_g(j).$$

By (1.6), for large enough n, $\lambda_f(n) < (1+\epsilon)c_f e^{A\sqrt{n}}/n^\alpha$. Thus for large enough n and $j \le \delta n$,

$$\lambda_f(n-j) < (1+\epsilon)c_f\frac{e^{A\sqrt{n-j}}}{(n-j)^\alpha}.$$

Thus by (3.4) and (3.6), this implies that for large n

$$\begin{aligned} S_\delta(n) &< (1+\epsilon)\frac{c_f}{n^\alpha}\sum_{j=0}^{\lfloor \delta n\rfloor} G\left(\frac{j}{n}\right)e^{A\sqrt{n-j}}\lambda_g(j) \\ &< (1+\epsilon)^2 c_f\frac{e^{A\sqrt{n}}}{n^\alpha}\sum_{j=0}^{\lfloor \delta n\rfloor}\lambda_g(j)e^{-\frac{Aj}{2\sqrt{n}}}. \end{aligned}$$

By the non-negativity of all of the terms,

$$\begin{aligned} S_\delta(n) &< (1+\epsilon)^2 c_f\frac{e^{A\sqrt{n}}}{n^\alpha}\sum_{j=0}^{\infty}\lambda_g(j)e^{-\frac{Aj}{2\sqrt{n}}} \\ &= (1+\epsilon)^2 c_f\frac{e^{A\sqrt{n}}}{n^\alpha} g\left(\frac{Ai}{4\pi\sqrt{n}}\right). \end{aligned} \tag{3.7}$$

Since $g = i^k E_k - i^k$, the modularity of E_k implies

$$g\left(\frac{Ai}{4\pi\sqrt{n}}\right) = \left(\frac{4\pi\sqrt{n}}{A}\right)^k E_k\left(\frac{4\pi\sqrt{n}i}{A}\right) - i^k.$$

Since $\lim_{n\to\infty} E_k\left(\frac{4\pi\sqrt{n}i}{A}\right) = 1$, for large enough n we have

$$g\left(\frac{Ai}{4\pi\sqrt{n}}\right) < (1+\epsilon)\left(\frac{4\pi\sqrt{n}}{A}\right)^k.$$

Combining this with (3.7) shows that for large enough n,

$$S_\delta(n) < (1+\epsilon)^3 c_f\frac{e^{A\sqrt{n}}}{n^{\alpha-k/2}}\left(\frac{4\pi}{A}\right)^k.$$

Since $\epsilon > 0$ was arbitrary, (3.5) follows immediately.

It remains to show that

$$\liminf_{n\to\infty}\frac{\lambda_{fg}(n)}{c_f\left(\frac{4\pi}{A}\right)^k\frac{e^{A\sqrt{n}}}{n^{\alpha-k/2}}} \geq 1. \tag{3.8}$$

Let $\epsilon > 0$ be given and set $\delta = n^{-1/3}$. By the non-negativity of the terms,

$$\lambda_{fg}(n) \geq \sum_{j=0}^{\lfloor \delta n\rfloor}\lambda_f(n-j)\lambda_g(j) = S_\delta(n).$$

By (1.6), for large enough n,

$$S_\delta(n) > (1-\epsilon)\frac{c_f}{n^\alpha}\sum_{j=0}^{\lfloor \delta n\rfloor} G\left(\frac{j}{n}\right)e^{A\sqrt{n-j}}\lambda_g(j).$$

By (3.3) and (3.4), for large enough n,

$$S_\delta(n) > (1-\epsilon)e^{-\frac{A\sqrt{n}\delta^2}{8(1-\delta)^{3/2}}}c_f\frac{e^{A\sqrt{n}}}{n^\alpha}\sum_{j=0}^{\lfloor\delta n\rfloor}e^{-\frac{Aj}{2\sqrt{n}}}\lambda_g(j).$$

In a moment we will prove the following.

Lemma 5. *For $\delta = n^{-1/3}$, we have*

$$\lim_{n\to\infty}\frac{\sum_{j=\lfloor\delta n\rfloor+1}^{\infty}\lambda_g(j)e^{-\frac{Aj}{2\sqrt{n}}}}{\sum_{j=0}^{\infty}\lambda_g(j)e^{-\frac{Aj}{2\sqrt{n}}}} = 0.$$

Since obviously

$$\sum_{j=0}^{\lfloor\delta n\rfloor}\lambda_g(j)e^{-\frac{Aj}{2\sqrt{n}}} = g\left(\frac{Ai}{4\pi\sqrt{n}}\right)\left(1-\frac{\sum_{j=\lfloor\delta n\rfloor+1}^{\infty}\lambda_g(j)e^{-\frac{Aj}{2\sqrt{n}}}}{\sum_{j=0}^{\infty}\lambda_g(j)e^{-\frac{Aj}{2\sqrt{n}}}}\right),$$

for large enough n Lemma 5 implies

$$S_\delta(n) > (1-\epsilon)^2e^{-\frac{A\sqrt{n}\delta^2}{8(1-\delta)^{3/2}}}c_f\frac{e^{A\sqrt{n}}}{n^\alpha}g\left(\frac{Ai}{4\pi\sqrt{n}}\right). \tag{3.9}$$

Since $g = i^kE_k - i^k$, the modularity of E_k implies

$$g\left(\frac{Ai}{4\pi\sqrt{n}}\right) = \left(\frac{4\pi\sqrt{n}}{A}\right)^k E_k\left(\frac{4\pi\sqrt{n}i}{A}\right) - i^k.$$

Since $\lim_{n\to\infty}E_k\left(\frac{4\pi\sqrt{n}i}{A}\right) = 1$, for large enough n we have

$$g\left(\frac{Ai}{4\pi\sqrt{n}}\right) > (1-\epsilon)\left(\frac{4\pi\sqrt{n}}{A}\right)^k.$$

Combining this with (3.9) shows that for large enough n,

$$S_\delta(n) > (1-\epsilon)^3e^{-\frac{A\sqrt{n}\delta^2}{8(1-\delta)^{3/2}}}c_f\left(\frac{4\pi}{A}\right)^k\frac{e^{A\sqrt{n}}}{n^{\alpha-k/2}}.$$

Since $\lim_{n\to\infty}\frac{\sqrt{n}\delta^2}{(1-\delta)^{3/2}} = \lim_{n\to\infty}\frac{n^{-1/6}}{(1-n^{-1/3})^{3/2}} = 0$, the inequality (3.8) follows immediately. It remains to prove the lemma.

Proof of Lemma 5. We first claim that for all integers $j \geq 1$ and all real $0 < \beta < 1$,

$$\sigma_{k-1}(j) \leq j^k \leq \left(\frac{k}{\beta}\right)^k e^{\beta j}.$$

On the one hand, if $j \le k/\beta$, then $j^k \le (k/\beta)^k \le (k/\beta)^k e^{\beta j}$. On the other hand, if $j > k/\beta$, then

$$j^k \le \left(\frac{k}{\beta}\right)^k e^{\beta j} \Leftrightarrow \left(\frac{\beta}{k}\right)^k \le \frac{e^{\beta j}}{j^k}.$$

Since the function $e^{\beta x}/x^k$ is non-decreasing for $x \ge k/\beta$, we know

$$\frac{e^{\beta j}}{j^k} \ge \frac{e^{\beta\left(\frac{k}{\beta}\right)}}{\left(\frac{k}{\beta}\right)^k} = \left(\frac{\beta}{k}\right)^k e^k > \left(\frac{\beta}{k}\right)^k.$$

This proves the claim.

Hence, for all $0 < \beta < 1$,

$$\begin{aligned} 0 < \frac{\sum\limits_{j=\lfloor \delta n\rfloor+1}^{\infty} \lambda_g(j) e^{-\frac{Aj}{2\sqrt{n}}}}{\sum\limits_{j=0}^{\infty} \lambda_g(j) e^{-\frac{Aj}{2\sqrt{n}}}} &\le \frac{\sum\limits_{j=\lfloor \delta n\rfloor+1}^{\infty} \sigma_{k-1}(j) e^{-\frac{Aj}{2\sqrt{n}}}}{\sum\limits_{j=1}^{\infty} e^{-\frac{Aj}{2\sqrt{n}}}} \\ &\le \left(\frac{k}{\beta}\right)^k \frac{\sum\limits_{j=\lfloor \delta n\rfloor+1}^{\infty} e^{\left(\beta-\frac{A}{2\sqrt{n}}\right)j}}{\sum\limits_{j=1}^{\infty} e^{-\frac{Aj}{2\sqrt{n}}}}. \end{aligned}$$

If $\beta - \frac{A}{2\sqrt{n}} < 0$, then the geometric series converge to give

$$\begin{aligned} \frac{\sum\limits_{j=\lfloor \delta n\rfloor+1}^{\infty} \lambda_g(j) e^{-\frac{Aj}{2\sqrt{n}}}}{\sum\limits_{j=0}^{\infty} \lambda_g(j) e^{-\frac{Aj}{2\sqrt{n}}}} &\le \left(\frac{k}{\beta}\right)^k \left(\frac{1-e^{-\frac{A}{2\sqrt{n}}}}{e^{-\frac{A}{2\sqrt{n}}}}\right) \left(\frac{e^{\left(\beta-\frac{A}{2\sqrt{n}}\right)(\lfloor \delta n\rfloor+1)}}{1-e^{\left(\beta-\frac{A}{2\sqrt{n}}\right)}}\right) \\ &\le \left(\frac{k}{\beta}\right)^k \left(\frac{1-e^{-\frac{A}{2\sqrt{n}}}}{e^{-\frac{A}{2\sqrt{n}}}}\right) \left(\frac{e^{\left(\beta-\frac{A}{2\sqrt{n}}\right)\delta n}}{1-e^{\left(\beta-\frac{A}{2\sqrt{n}}\right)}}\right). \end{aligned}$$

Choose $\beta = \frac{A}{2n^{3/2}}$. Now

$$\frac{\sum\limits_{j=\lfloor \delta n\rfloor+1}^{\infty} \lambda_g(j) e^{-\frac{Aj}{2\sqrt{n}}}}{\sum\limits_{j=0}^{\infty} \lambda_g(j) e^{-\frac{Aj}{2\sqrt{n}}}} \le \left(\frac{2k}{A}\right)^k \left(\frac{1-e^{-\frac{A}{2\sqrt{n}}}}{1-e^{-\frac{A}{2\sqrt{n}}\left(1-\frac{1}{n}\right)}}\right) \left(\frac{n^{3k/2}}{e^{\frac{A\delta\sqrt{n}}{2}\left(1-\frac{1}{n}\right)}}\right) e^{\frac{A}{2\sqrt{n}}}.$$

Substitute $\delta = n^{-1/3}$ to get

$$\frac{\sum_{j=\lfloor \delta n \rfloor+1}^{\infty} \lambda_g(j) e^{-\frac{Aj}{2\sqrt{n}}}}{\sum_{j=0}^{\infty} \lambda_g(j) e^{-\frac{Aj}{2\sqrt{n}}}} \le \left(\frac{2k}{A}\right)^k \left(\frac{1-e^{-\frac{A}{2\sqrt{n}}}}{1-e^{-\frac{A}{2\sqrt{n}}(1-\frac{1}{n})}}\right) \left(\frac{n^{3k/2}}{e^{\frac{An^{1/6}}{2}(1-\frac{1}{n})}}\right) e^{-\frac{A}{2\sqrt{n}}}$$

$$\to \left(\frac{2k}{A}\right)^k \cdot 1 \cdot 0 \cdot 1,$$

as $n \to \infty$. This proves the lemma. □

Acknowledgments

The authors would like to thank the referee for various interesting remarks: Theorem 1 should also follow from the development of Poincare series (see [6]). There will of course be problems with convergence for low weights, including zero. The referee also remarked that our methods have a similar flavor to Tauberian theorems (see, for example, [5]). In particular, our proof of Theorem 3 relies on the asymptotic for $E_k(q)$ as $q \to 1$, which we deduce via modularity.

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