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An all-purpose Erdös-Kac theorem

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Abstract

In a seminal paper of 1917, Hardy and Ramanujan showed that the normal number of prime factors of a random natural number n is $\log \log n$. Their paper is often seen as inspiring the development of probabilistic number theory in that it led Erdös and Kac to discover, in 1940, a Gaussian law implied by their work. In this paper, we derive an all-purpose Erdös-Kac theorem that is applicable in diverse settings. In particular, we apply our theorem to show the validity of an Erdös-Kac type theorem for the study of the number of prime factors of sums of Fourier coefficients of Hecke eigenforms.

Mathematics Subject Classification Primary 11K65 · 11N75; Secondary 11N37

1 Introduction

Let $\omega(m)$ denote the number of distinct prime divisors of a natural number m. In 1917, G. H. Hardy and S. Ramanujan [13] proved that if we fix $\epsilon > 0$, the number of $m \leq n$ for which the inequality

$$|\omega(m) - \log \log m| > (\log \log m)^{1/2+\epsilon}$$

holds is $o(n)$ as n tends to infinity. Thus, a random number m "usually" has $\log \log m$ distinct prime divisors in this sense. More precisely, we say that the "normal order" of $\omega(m)$ is $\log \log m$. The Hardy-Ramanujan paper was quite lengthy and involved estimates for the

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number of integers $\le n$ with exactly k prime divisors. In 1934, Turán [38] gave an extremely simple, short and elegant proof of their theorem by showing that

$$\sum_{m \le n} (\omega(m) - \log\log m)^2 = O(n \log\log n).$$

This result suggested to M. Kac [15] that there may be a probabilistic perspective of the Hardy-Ramanujan theorem. Thus, in 1940, Erdös and Kac wrote their celebrated paper [7] in which they derived a Gaussian law for the number of prime factors of a random natural number. They showed the following. On the set of natural numbers, let P_n be the probability measure that assigns $1/n$ for each of 1, 2, ..., n and zero elsewhere. Then

$$\lim_{n \to \infty} P_n \left(m : \frac{\omega(m) - \log\log m}{\sqrt{\log\log m}} \le x \right) = \Phi(x),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

is the normal distribution. The Erdös-Kac theorem gave birth to probabilistic number theory and an exposition of these developments can be found in the monograph of Kubilius [16].

In 1984, the first two authors [25] initiated a study of the normal number of prime factors of Fourier coefficients of Hecke eigenforms. We illustrate their results in the special case of the Ramanujan τ-function. Recall that this function is defined by the power series expansion of the infinite product:

$$q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

If we set $q = e^{2\pi i z}$, with z in the upper half-plane, this is a modular form of weight 12 for the full modular group. Ramanujan [33] conjectured that $\tau(n)$ is a multiplicative function of n and that for primes p, $|\tau(p)| \le 2p^{11/2}$. The first conjecture was proved by Mordell [24] in 1919. The latter conjecture was proved by Deligne [4] in 1974 as a consequence of his proof of the Weil conjectures. An important ingredient of Deligne's proof of the Ramanujan conjecture involves the existence of an ℓ-adic representation ρ_ℓ of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that

$$\rho_\ell : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_\ell), \qquad \mathrm{tr}\, \rho_\ell(\sigma_p) = \tau(p), \qquad p \ne \ell,$$

where $\mathbb{Z}_\ell$ denotes the ring of ℓ-adic integers, and σ_p is the Frobenius automorphism attached to p. The details of this construction can be found in [3]. The existence of this representation opens the door for an arithmetical study of $\tau(p)$ using methods of analytic number theory. More precisely, the first two authors in [25] and [26] used the Chebotarev density theorem and a quasi-generalized Riemann hypothesis to show that if $\tau(p) \ne 0$, the normal number of prime factors of $\tau(p)$ is $\log\log p$. More precisely, if we let $\pi(n)$ be the number of primes $p \le n$ and denote by P_n^* the probability measure which assigns $1/\pi(n)$ to each prime $p \le n$, then

$$\lim_{n \to \infty} P_n^* \left(p : \frac{\omega(\tau(p)) - \log\log p}{\sqrt{\log\log p}} \le x \right) = \Phi(x).$$

Deligne's result is more general than what we have stated here. He constructs Galois representations for each Hecke eigenform of arbitrary weight $k \ge 2$ with similar properties. In the intervening years, several authors have extended Deligne's work in various directions. First, they identified the image. Second, they attached similar representations to two normalised Hecke eigenforms f and g. For instance, Ribet [34] in the case of level one, Momose [23] in the case of higher level, and more recently Loeffler [21] have computed the image of the Galois representation

$$\rho_f \times \rho_g : G_{\mathbb{Q}} \to GL_2(\widehat{\mathbb{Z}}) \times GL_2(\widehat{\mathbb{Z}}),$$

such that

$$\operatorname{tr} \rho_f(\sigma_p) = a_f(p), \qquad \rho_g(\sigma_p) = a_g(p),$$

where $a_f(p), a_g(p)$ are the p-th Fourier coefficients of f and g respectively. Here $\widehat{\mathbb{Z}}$ is the Prüfer ring which has the canonical decomposition

$$\widehat{\mathbb{Z}} = \prod_{\ell} \mathbb{Z}_\ell.$$

Using these results, we showed in a recent paper [27] that under a quasi-generalized Riemann hypothesis, the normal number of prime factors of $a_f(p)+a_g(p)$ is $\log\log p$. The framework was general and can be applied to a wide variety of sequences. The quasi-generalized Riemann hypothesis referred to is the assertion that there exists a $\delta \in (1/2, 1)$ such that the Dedekind zeta function of any algebraic number field has no zero in the region $\Re(s) > \delta$. The assertion is substantially weaker than the generalized Riemann hypothesis.

In this paper, we want to establish the analogue of the Erdös-Kac theorem studying the number of prime factors of $a_f(p) + a_g(p)$. To this end, we will axiomatize our approach so that it is applicable in a wider context. As a consequence of this axiomatic treatment, we will prove in particular:

Theorem 1 *Let f and g be two normalised Hecke eigenforms of weight 2 with integer coefficients. For primes p, let the p-th Fourier coefficient of f and g be denoted $a_f(p)$ and $a_g(p)$ respectively. Assuming a quasi-generalized Riemann hypothesis for Dedekind zeta functions,*

$$\lim_{n\to\infty} P_n^*\left(p : \frac{\omega(a_f(p) + a_g(p)) - \log\log p}{\sqrt{\log\log p}} \le x\right) = \Phi(x).$$

Unconditionally, we have the following:

Theorem 2 *For any fixed $C > 0$, let $\omega^\dagger(n)$ be the number of prime divisors of n less than $(\log n)^C$. Then,*

$$\lim_{n\to\infty} P_n^*\left(p : \frac{\omega^\dagger(a_f(p) + a_g(p)) - \log\log\log p}{\sqrt{\log\log\log p}} \le x\right) = \Phi(x).$$

Our theorems will follow from a very general "all-purpose" Erdös-Kac theorem that has wide applicability, as we will show in later sections. We say a sequence of numbers a_m has *polynomial growth* if there exists an $A > 0$ such that $a_m = O(m^A)$.

Theorem 3 *Let S be an infinite subset of the natural numbers. For $m \in S$, let a_m be a sequence of non-zero integers with polynomial growth. Let $S(n)$ be the number of $m \le n$ with $m \in S$ and for each d, let $S(n, d)$ be the number of $m \le n$ with $m \in S$ such that $d|a_m$. Let P_n^S be the probability distribution defined on $[1, n]$ with $P_n^S(a) = 1/S(n)$ if $a \in S$ and zero otherwise. Suppose there is a multiplicative function $\delta(d)$ and an $\eta > 0$ such that for any $B > 0$, and as $n \to \infty$, one has*

$$\sum_{d<n^\eta} \mu^2(d)|S(n, d) - \delta(d)S(n)| \ll S(n)\mu_n^{-B}, \tag{1}$$

where

$$\mu_n = \sum_{\ell \le n} \delta(\ell) \to \infty,$$

as $n \to \infty$. *Then,*

$$\lim_{n\to\infty} P_n^S \left(m : \frac{\omega(a_m) - \log\log m}{\sqrt{\log\log m}} \le x \right) = \Phi(x).$$

Condition (1) in the above theorem thus encapsulates the essential condition needed to deduce an Erdös-Kac type theorem for a general sequence. The condition reminds us of criteria that often appear in sieve theory. But unlike the sieve theory setting, we have in all our applications, μ_n asymptotic to $\log\log n$. In other words, we require saving by arbitrary powers of $\log\log n$ instead of saving by arbitrary powers of $\log n$. We will elaborate on this further in Sects. 5 and 6.

2 An application of the central limit theorem

The normal distribution Φ seems to be ubiquitous in mathematics. It is determined by its moments. More precisely, if we let

$$M_r := \int_{-\infty}^{\infty} x^r d\Phi(x),$$

be the r-th moment, and distribution functions F_n are such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x^r dF_n(x) = M_r, \qquad r = 1, 2, ...,$$

then $F_n(x) \to \Phi(x)$ for each x. This is the basis of the method of moments in probability theory (see (b) on p. 269 of [9]). This is one of the ingredients we will use in the following. Another ingredient is Lindeberg's version of the central limit theorem (see Theorem 3 on page 262 of [9]). One implication of this version is that if the X_k's are **bounded** independent random variables with mean zero such that

$$s_n^2 := \mathrm{Var}(X_1 + \cdots + X_n) \to \infty$$

as $n \to \infty$, then $(X_1 + \cdots + X_n)/s_n$ converges to the normal distribution with mean zero and variance 1 (see (e) on page 264 of [9]). We begin with an application of this version of the central limit theorem which will animate our later discussion. We modify an arrangement of Billingsley [1]. For each prime ℓ, let X_ℓ be independent random variables on some probability space satisfying

$$Pr(X_\ell = 1) = \delta(\ell), \qquad Pr(X_\ell = 0) = 1 - \delta(\ell).$$

Fix any $\epsilon > 0$ and let

$$\alpha_n = n^\epsilon, \qquad \epsilon \in (0, 1).$$

Consider the sum

$$S_n = \sum_{\ell < \alpha_n} X_\ell.$$

Using the independence of the X_ℓ's, the mean and variance of S_n, denoted μ_n and σ_n^2 respectively, are easily computed to be:

$$\mu_n = \sum_{\ell \le \alpha_n} \delta(\ell), \qquad \sigma_n^2 = \sum_{\ell \le \alpha_n} \delta(\ell)\,(1 - \delta(\ell))\,.$$

We have by an application of the central limit theorem (as noted above, see (e) on page 264 of [9]) that the distribution of

$$\frac{S_n - \mu_n}{\sigma_n}$$

converges to the normal distribution Φ. In particular, by our remarks above regarding the moments of the normal distribution, we have

$$\lim_{n\to\infty} E[(S_n - \mu_n)^r / \sigma_n^r] = \int_{-\infty}^{\infty} x^r \, d\Phi(x), \tag{2}$$

where E denotes the usual expectation function. This result will be used below in our derivation of a general Erdös-Kac theorem.

It is also convenient to observe that if $\ell_1, ..., \ell_k$ are distinct primes and $a_1, ..., a_k$ are positive integers, the expectation

$$E(X_{\ell_1}^{a_1} \cdots X_{\ell_k}^{a_k})$$

equals $\delta(\ell_1) \cdots \delta(\ell_k)$. This fact will also be used in the later sections.

3 A general Erdös-Kac theorem

We consider a sequence a_m of non-zero integers defined on an infinite subset S of the natural numbers $\mathbb{N}$. Let $S(n)$ be the number of $m \in S$ with $m \le n$. We assume the following four axioms are satisfied.

(a) There is an $A > 0$ such that $a_m = O(m^A)$;

(b) For each natural number d, write $S(n, d)$ to be the number of $m \le n$ with $m \in S$ such that $d | a_m$. For any fixed squarefree d, we suppose that there is a multiplicative function $d \mapsto \delta(d)$ with $0 \le \delta(d) \le 1$ such that

$$S(n, d) = \delta(d) S(n) + o(S(n)),$$

as n tends to infinity;

(c) We assume

$$\limsup_{\ell\to\infty} \delta(\ell) < 1,$$

where the limit is over primes ℓ;

(d) For any $B > 0$, there exists an $\eta = \eta_B > 0$ such that

$$\sum_{d < n^\eta} \mu^2(d)\, |S(n, d) - \delta(d) S(n)| \ll \frac{S(n)}{\mu_n^B},$$

and

$$\mu_n := \sum_{\ell \le \alpha_n} \delta(\ell) \to \infty \text{ as } n \to \infty.$$

We will use these $\delta(\ell)$'s to construct the random variables X_ℓ's discussed in the previous section. Let us note that axiom (c) allows us to deduce that

$$\sigma_n^2 := \sum_{\ell \leq \alpha_n} \delta(\ell)(1-\delta(\ell)) \gg \sum_{\ell \leq \alpha_n} \delta(\ell) = \mu_n, \tag{3}$$

a fact that will play a crucial role in our estimates. In all our applications, $\delta(\ell)$ will be asymptotic to $1/\ell$ so that μ_n is asymptotic to $\log\log n$. We have put it in this form so as to facilitate a more general setting.

With these axioms in place, we will prove the following theorem by an application of the central limit theorem. Given a subset S of $\mathbb{N}$, we introduce the following probability measures. On the set of positive integers, let P_n^S be the probability measure that puts mass $1/S(n)$ at each of $m \leq n$ and $m \in S$ and zero elsewhere.

Theorem 4 *Suppose that we have a sequence a_m of non-zero integers defined on an infinite subset S of the natural numbers satisfying the axioms (a), (b), (c) and (d). Then*

$$\lim_{n\to\infty} P_n^S \left(m \in S : \frac{\omega(a_m) - \mu_m}{\sigma_m} \leq x \right) = \Phi(x).$$

Our strategy is to introduce cognate random variables Y_ℓ (with ℓ prime) that mimic X_ℓ constructed above. Define $Y_\ell(m) = 1$ if ℓ divides a_m and zero otherwise. We set

$$\nu_n(m) := \sum_{\ell \leq \alpha_n} Y_\ell(m).$$

Thus, $\nu_n(m)$ is the number of prime divisors of a_m less than or equal to α_n. Since $\alpha_n = n^\epsilon$, and a_m has polynomial growth, we see that the number of prime divisors greater than α_n is bounded. Consequently, we see that $\nu_n(m) = \omega(a_m) + O(1)$. To prove the theorem, we apply the method of moments. Thus, our goal is to show that for each fixed $r \geq 1$,

$$\lim_{n\to\infty} E[(\nu_n - \mu_n)^r/\sigma_n^r] = \int_{-\infty}^{\infty} x^r d\Phi(x).$$

In view of (2), it suffices to show that

$$\lim_{n\to\infty} \left\{ E[(\nu_n - \mu_n)^r/\sigma_n^r] - E[(S_n - \mu_n)^r/\sigma_n^r] \right\} = 0.$$

A simple application of the binomial theorem together with (3) shows that it suffices to prove that for every fixed $j \leq r$,

$$\lim_{n\to\infty} |E[\nu_n^j] - E[S_n^j]| \mu_n^{r/2} = 0. \tag{4}$$

Our axiomatic framework will facilitate the proof of this fact which we give in the next section.

4 Proof of Theorem 4

Our goal is to show (4) for every $r \geq 1$. In axiom (d), we choose $B > r$ so that η exists. We will choose $\epsilon < \eta/r$. On the one hand, S_n is the sum of the random variables X_ℓ for $\ell \leq \alpha_n$. On the other hand, ν_n is the sum of the variables Y_ℓ. An application of the multinomial theorem shows that

$$E[\nu_n^j] - E[S_n^j] = \sum_{j_1,\ldots,j_u} \binom{j}{j_1,\ldots,j_u} \sum_{\ell_1,\ldots,\ell_u} \left(E[X_{\ell_1}^{j_1} \cdots X_{\ell_u}^{j_u}] - E[Y_{\ell_1}^{j_1} \cdots Y_{\ell_u}^{j_u}] \right),$$

where the $j_i \leq j$ and $u = \pi(\alpha_n)$ and the ℓ_i are prime numbers $\leq \alpha_n$. As X_ℓ and Y_ℓ take values either 0 or 1, we see $X_\ell^t = X_\ell$ and $Y_\ell^t = Y_\ell$ for any t, so that the innermost summand is a term of the form

$$E[X_{\ell_1} \cdots X_{\ell_k}] - E[Y_{\ell_1} \cdots Y_{\ell_k}],$$

with $\ell_1 < \ell_2 < \cdots \ell_k \leq \alpha_n$ and $k \leq u$. We have already noted that $E[X_{\ell_1} \cdots X_{\ell_k}] = \delta(\ell_1) \cdots \delta(\ell_k)$. Also,

$$E[Y_{\ell_1} \cdots Y_{\ell_k}] = \frac{S(n, \ell_1 \cdots \ell_k)}{S(n)}$$

so that our innermost summand is precisely

$$\delta(\ell_1 \cdots \ell_k) - \frac{S(n, \ell_1 \cdots \ell_k)}{S(n)},$$

because $\ell_1 \cdots \ell_k$ is squarefree. As each $\ell_i < \alpha_n = n^\epsilon$, we have $\ell_1 \cdots \ell_k < \alpha_n^k$ and as $k \leq j \leq r$, we have $\ell_1 \cdots \ell_k < \alpha_n^r \leq n^\eta$. Moreover, each squarefree number $\ell_1 \cdots \ell_k$ can occur with multiplicity at most r^r times because we require

$$j_1 + \cdots + j_u = j \leq r,$$

and so each j_i has at most $r + 1$ choices and $u \leq r$. By our axiom, and our choice of ϵ, we see that

$$\sum_{\ell_1, \ldots, \ell_u} \left| E[X_{\ell_1}^{j_1} \cdots X_{\ell_u}^{j_u}] - E[Y_{\ell_1}^{j_1} \cdots Y_{\ell_u}^{j_u}] \right| \ll \mu_n^{-B},$$

where the implied constant depends only on r. Since $B > r$, we see that (4) is now proved.

5 From ω to Ω

Let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity. In probabilistic number theory, one often finds that the Erdös-Kac theorem also holds for Ω. We want to ensure that a similar phenomenon persists in our general axiomatic framework. To this end, we make the following hypothesis:

$$\sum_{m \leq n, m \in S} (\Omega(a_m) - \omega(a_m)) = O(S(n)). \tag{5}$$

It is clear that we can restrict our attention to prime divisors less than α_n. Following the strategy suggested by Billingsley [1], we define $Y_\ell'(m)$ to be the power of the prime ℓ dividing a_m and set

$$\nu_n'(m) = \sum_{\ell \leq \alpha_n} Y_\ell'(m).$$

Then, $\nu_n'(m)$ is the number of prime divisors of a_m less than α_n counted with multiplicity. Then, (5) is equivalent to $E(\nu_n' - \nu_n) = O(1)$. This condition is equivalent to

(e) $\displaystyle\sum_{\ell \leq \alpha_n} \sum_{k=2}^{\infty} S(n, \ell^k) = O(S(n)).$

Indeed, we must ensure that

$$\sum_{\ell \leq \alpha_n} \sum_{k=2}^{\infty} k \left(S(n, \ell^k) - S(n, \ell^{k+1}) \right) = O(S(n)). \tag{6}$$

Since

$$\sum_{k=2}^{\infty} k \left(S(n, \ell^k) - S(n, \ell^{k+1}) \right) = \sum_{k=2}^{\infty} k S(n, \ell^k) - \sum_{k=3}^{\infty} (k-1) S(n, \ell^k),$$

we see that our condition is equivalent to (e) as claimed. One more remark is worth underlining. The inner sum in (e) is a finite sum since our sequence has polynomial growth and $S(n, \ell^k) = 0$ for $k \gg \log n$. We have the following theorem:

Theorem 5 *If in addition to the conditions of Theorem 4, we have also (e), then*

$$\lim_{n\to\infty} P_n^S \left(m \in S : \frac{\Omega(a_m) - \mu_m}{\sigma_m} \leq x \right) = \Phi(x).$$

There are several ways of deducing this theorem. A first way is to follow the strategy of Billingsley [1] who cites a theorem in Feller [9]. To explain this theorem, we say a sequence of random variables $(X_n)_{n\geq 1}$ converges in probability to zero if for each $\epsilon > 0$, $P(|X_n| > \epsilon) \to 0$ as n tends to infinity. Then, Lemma 2 on page 254 of [9] says that if X_n converges in probability to zero and if $(Y_n)_{n\geq 1}$ has distribution Φ, then $(X_n + Y_n)_{n\geq 1}$ also has distribution Φ. If we apply this to our context, then condition (5) ensures that $\nu_n' - \nu_n$ tends to zero in probability and we can then apply the theorem from [9] to deduce the result.

On the other hand, one can approach this directly via standard methods of analytic number theory. Indeed, Theorem 4 tells us that

$$\#\left\{ m \leq n, m \in S : \frac{\omega(a_m) - \mu_m}{\sigma_m} \leq x \right\} = S(n)\Phi(x) + o(S(n)).$$

But (5) tells us that

$$|\Omega(a_m) - \omega(a_m)| > \sigma_n^{1/2}$$

happens for at most $O(S(n)/\sigma_n^{1/2})$ which is $o(S(n))$ because σ_n tends to infinity by (3). If we discard these numbers from our enumeration, we have $\Omega(a_m) - \omega(a_m) \leq \sigma_n^{1/2}$ for the remaining, and $1/\sigma_n^{1/2} \to 0$ as $n \to \infty$, we deduce

$$\#\left\{ m \leq n, m \in S : \frac{\Omega(a_m) - \mu_m}{\sigma_m} \leq x \right\} = S(n)\Phi(x) + o(S(n)).$$

One can also go in the other direction. If we had an Erdös-Kac theorem for $\Omega(a_m)$ and condition (e) is satisfied, then $\omega(a_m)$ also satisfies an Erdös-Kac law. The advantage of working with Ω instead of ω, in certain cases, emanates from its' complete additivity.

6 First applications

As pointed out by Billingsley [1], sieve theory can be dispensed with in the derivation of the Erdös-Kac theorem, though it was an essential ingredient in the original paper of Erdös and Kac [7]. We enumerate here several immediate applications of our formalism.

(A) *The classical Erdös-Kac theorem* [7] is an immediate consequence. Indeed, in this case $\delta(\ell) = 1/\ell$ and

$$S(n, d) = \frac{n}{d} + O(1),$$

so that the condition of our Theorem 3 is valid for any $\eta < 1$. Since

$$\sum_{\ell \le \alpha_n} \sum_{k=2}^{2\log n} S(n, \ell^k) = O(n),$$

we see from Theorem 5 that the Erdös-Kac theorem holds for $\Omega(n)$ also.

(B) Let $f(n)$ be an irreducible polynomial of degree t, with integer coefficients. Then

$$\lim_{n\to\infty} P_n \left(m : \frac{\omega(f(m)) - \log\log m}{\sqrt{\log\log m}} \le x \right) = \Phi(x).$$

To deduce this from our general theorem, we need only observe that for each squarefree d,

$$S_f(n, d) := \#\{m \le n : f(m) \equiv 0 \,(\mathrm{mod}\, d)\} = \frac{\rho(d)}{d} n + O(\rho(d)),$$

where $\rho(d)$ is the number of solutions of the congruence $f(m) \equiv 0 (\mathrm{mod}\, d)$. By the Chinese remainder theorem, $\rho(d)$ is a multiplicative function and as

$$\sum_{d \le x} \mu^2(d)\rho(d) \ll \sum_{d \le x} t^{\omega(d)} \ll x(\log x)^{t-1},$$

we see that the condition of Theorem 3

$$\sum_{d < n^\eta} |S_f(n, d) - \frac{\rho(d)}{d} n| \ll \frac{n}{(\log\log n)^B},$$

is satisfied with any $\eta < 1$ and any $B > 0$. To pass from ω to Ω, we need to check that (e) holds for this sequence. But this is clear since $\alpha_n = n^\epsilon$ and $\rho(d) = O(d^\epsilon)$. Indeed, we have for a suitable constant C,

$$\sum_{\ell < \alpha_n} \sum_{k=2}^{C\log n} S(n, \ell^k) \ll \sum_{\ell < \alpha_n} \sum_{k=2}^{C\log n} \left(\frac{\rho(\ell^k)}{\ell^k} n + O(\rho(\ell^k)) \right) \ll S(n).$$

(C) *Halberstam's theorem* [11] states that if a is a fixed natural number $\neq 0$, and p is a prime, then the normal number of prime factors of $p + a$ is $\log\log p$. Moreover, an analogue of the Erdös-Kac theorem holds. If S is the sequence of prime numbers and $a_p = p + a$ for some fixed integer $a \neq 0$, then (b) is a consequence of the prime number theorem for arithmetic progressions with $\delta(\ell) = 1/(\ell - 1)$. Condition (d) is a consequence of the Bombieri-Vinogradov theorem (see for example, chapter 9 of [2]). Thus, with Halberstam [11], we deduce that

$$\lim_{n\to\infty} \frac{1}{\pi(n)} \# \left\{ p \le n : \frac{\omega(p + a) - \log\log p}{\sqrt{\log\log p}} \le x \right\} = \Phi(x).$$

It is interesting to note that the normal order of $\omega(p + a)$ had been established earlier by Erdös [6], using Brun's sieve. When Halberstam proved his theorem in 1955, he did not have the Bombieri-Vinogradov theorem. The use of the latter theorem was cleverly circumvented by Halberstam via technical devices from sieve theory. A small variation of this argument shows that one can replace ω by Ω where $\Omega(n)$ is the number of prime factors of n counted with multiplicity. Indeed, by our formalism condition (e) amounts to showing that for a suitable constant C,

$$\sum_{\ell \leq \alpha_n} \sum_{k=2}^{C \log n} \pi(n, \ell^k) = O(\pi(n)),$$

where $\pi(n, d)$ is the number of primes $p \leq n$ such that $d|p+a$. We split the double sum into two parts: $\ell^k < \sqrt{n}$ and $\ell^k > \sqrt{n}$. On the first part, we use the Brun-Titchmarsh inequality(see for example, pages 125-127 of [2]). This states that

$$\pi(n, d) \leq \frac{2n}{\phi(d) \log(n/d)}, \qquad d < n,$$

and applying this, we deduce the first part is $O(\pi(n))$. For the second part, we use the trivial bound

$$\pi(n, d) \leq \frac{n}{d} + 1 \ll \sqrt{n}.$$

As the number of summands is at most $O(\alpha_n \log n)$, the contribution of the second part from the above estimate is seen to be negligible. Thus, we deduce the Erdös-Kac theorem for $\Omega(p+a)$ also.

(D) More generally, one can consider an irreducible polynomial f with integer coefficients and of degree t and show that for primes p, $\omega(f(p+a))$ obeys an Erdös-Kac law, since again, the crucial condition (d) is a consequence of the Bombieri-Vinogradov theorem. Indeed, using the notation of (B) above, we see that for squarefree d, the number of primes $p \leq n$ such that $f(p+a)$ is divisible by d is the same as the number of primes $p \leq n$ that lie in $\rho(d)$ arithmetic progressions mod d. We leave the details to the reader. This theorem was also proved by Halberstam [12] in 1956 without the use of the Bombieri–Vinogradov theorem but by an intricate application of sieve theory. A similar Erdös-Kac law holds for $\Omega(f(p+a))$ also. The verification of condition (e) in this context is enabled again by the use of the Brun–Titchmarsh inequality.

7 Further applications

For applications discussed in this section, we will need to remind the reader of the Chebotarev density theorem. The following versions of the effective Chebotarev density theorem derived in [17], [35] and [28] will be used at various places in this paper and so we record it below for ease of reference.

Proposition 6 *Let L/K be a finite Galois extension of number fields with group G. Let C be a subset of G stable under conjugation. Denote by n_K the degree $[K : \mathbb{Q}]$, d_K the discriminant of K and $\pi_C(x)$ the number of prime ideals $\mathfrak{p}$ of K unramified in L, with norm less than or equal to x satisfying $\sigma_{\mathfrak{p}} \in C$ (where $\sigma_{\mathfrak{p}}$ is the Frobenius automorphism of $\mathfrak{p}$ attached to L/K). Let $P(L/K)$ be the set of rational primes p for which there is a prime ideal $\mathfrak{p}$ in K which ramifies in L and $\mathfrak{p}|p$. Set*

$$M(L/K) := [L : K]|d_K|^{1/n_K} \prod_{p \in P(L/K)} p.$$

Assuming the Riemann hypothesis holds for the Dedekind zeta function $\zeta_L(s)$, we have

(a)

$$\pi_C(x) = \frac{|C|}{|G|}\pi(x) + O\left(|C|x^{1/2}n_K \log(M(L/K)x)\right).$$

(b) *If in addition all the Artin L-series attached to irreducible representations of G are holomorphic at $s \neq 1$, then*

$$\pi_C(x) = \frac{|C|}{|G|}\pi(x) + O\left(|C|^{1/2}x^{1/2}n_K \log(M(L/K)x)\right).$$

Some remarks are in order. The difference between the two versions is in the error term where the appearance of $|C|$ in the first version is improved to $|C|^{1/2}$ in the second. Lagarias and Odlyzko [17] first obtained a variation of (a) of the effective Chebotarev density theorem assuming the generalized Riemann hypothesis. This was further refined by Serre [35] and the version in (a) is due to him. The improvement (b) arising from the additional assumption of the Artin holomorphy conjecture was derived by the authors in [28] and the form we have written down is on page 266 of that paper. In particular, if L/K is abelian, the Artin holomorphy conjecture holds and so the improved version in (b) is valid on the assumption of the generalized Riemann hypothesis alone. If we do not use the full generalized Riemann hypothesis, but only the weaker quasi-generalized Riemann hypothesis, that for some $\delta_1 > 0$, the Dedekind zeta function has no zeros to the right of $\Re(s) > \delta_1$, then the exponent $1/2$ in the error term can be replaced by δ_1. We will use these remarks below.

Regarding the quantity $M(L/K)$ that appears in Proposition 6, there are several ways of estimating it. For example, when L/K is Galois, using the estimates provided on page 259 of [28] or on page 129 of [35], we have

$$\log M(L/K) \ll \log[L:\mathbb{Q}] + \sum_{p \in P(L/K)} \log p$$

where the implied constant is absolute. We will also be implicitly using this result in various estimations below.

(a) *The Murty-Saidak theorem*: In [32], the authors prove the following theorem. Let a be a natural number greater than 1. For each prime p coprime to a, let $f_a(p)$ be the order of a mod p. Then, assuming a quasi-generalized Riemann hypothesis, they show that

$$\lim_{n\to\infty} \frac{1}{\pi(n)} \# \left\{ p \leq n : \frac{\omega(f_a(p)) - \log\log p}{\sqrt{\log\log p}} \leq x \right\} = \Phi(x).$$

This can be deduced from our formalism by an application of the (conditional) Chebotarev density theorem. We will establish the theorem with Ω instead of ω and the Murty–Saidak theorem then follows from our remarks made at the end of Sect. 5. Indeed, axiom (a) is clear. To verify axioms (b), (c), and (d), we need to study the Kummer extensions

$$L_d = \mathbb{Q}(\zeta_d, a^{1/d}),$$

where ζ_d denotes a primitive d-th root of unity. Since $f_a(p)|(p-1)$, it has polynomial growth. If we write $i_a(p)$ to be the index of the group generated by a in $\mathbb{F}_p^*$, then $f_a(p)i_a(p) = p - 1$. Then, $\Omega(f_a(p)) + \Omega(i_a(p)) = \Omega(p-1)$. For this sequence $f_a(p)$, we have $S(n,d) = \pi(n,d,1) - \pi_d(n)$, where $\pi_d(n)$ is the number of primes splitting completely in L_d. By the Chebotarev density theorem,

$$\pi_d(n) \sim \frac{\pi(n)}{d\phi(d)},$$

as n tends to infinity. A quasi-generalized Riemann hypothesis gives us an error term of $O(n^{\theta})$ with $\theta < 1$ uniformly for $d < n^{(1-\theta)/2}$. In any case, the conditions (b), (c) and (d) of Theorem 4 are verified. Finally, condition (e) is a consequence of the Brun-Titchmarsh inequality.

(b) *The Murty-Murty theorem*: In [26], the authors derived an Erdös-Kac law for the number of prime divisors of the Ramanujan τ-function assuming a quasi-generalized Riemann hypothesis. Without going into too much detail, the study of the prime divisors of $\tau(p)$ with p prime, is enabled by the ℓ-adic representation combined with the Chebotarev density theorem, as described in our introduction. However, before injecting this sequence into our axiomatic framework, we need to remove those primes p for which $\tau(p) = 0$. It is a famous conjecture of Lehmer [19] that this never happens, however, we are far from proving this. One can estimate the number of such primes $\leq n$ to be $O(n^{\theta})$ with $\theta < 1$, under a quasi-generalized Riemann hypothesis. This is easy to derive following a mild variation in the argument of section 4 of [28]. In other words, the number of such primes is negligible.
Here is a brief description of how to apply the Chebotarev density theorem in such contexts. As indicated on page 255 of [28], we begin with the ℓ-adic representation

$$\rho_{\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}_{\ell})$$

which has the property that for primes coprime to ℓ, the characteristic polynomials of the Frobenius automorphism σ_p is $T^2 - \tau(p)T + p^{11}$. Reducing this representation mod ℓ produces a finite extension $K_{\ell}/\mathbb{Q}$ which is the fixed field of the kernel of the reduction (mod ℓ) map. It is to these finite extensions to which we apply the Chebotarev density theorem.
Indeed, the field K_{ℓ} is ramified only at ℓ and as indicated on page 270 of [28], we apply the theorem of Ribet [34] to write that the Galois group of $K_{\ell}/\mathbb{Q}$ is

$$G_{\ell} := \{g \in GL_2(\mathbb{F}_{\ell}) : \det g \in (\mathbb{F}_{\ell}^{*})^{11}\}.$$

Thus primes p for which $\tau(p) \equiv 0 \pmod{\ell}$ correspond to σ_p whose trace is zero in $GL_2(\mathbb{F}_{\ell})$ and the set of such elements is easily seen to be of size $\asymp \ell^3$. Clearly,

$$\#\{p \leq n : \tau(p) = 0\} \leq \#\{p \leq n : \tau(p) \equiv 0 \pmod{\ell}\},$$

for any prime ℓ. By the effective Chebotarev density theorem as stated in (a) of Theorem 6, this last quantity is bounded by

$$\frac{\pi(n)}{\ell} + O(\ell^3 n^{1/2} \log \ell n), \tag{7}$$

assuming GRH. If, in addition, one assumes the Artin holomorphy conjecture, the ℓ^3 term in the error can be replaced by $\ell^{3/2}$. In any case, we can choose $\ell = [n^{1/8}/\log n]$ to get a bound of $O(n^{7/8})$ for the number of primes $p \leq n$ for which $\tau(p) = 0$. This can be improved in a variety of ways as discussed in [28], but this is not essential for our discussion here.
Removing primes $p \leq n$ for which $\tau(p) = 0$ from our discussion, we find that all the conditions of Theorem 4 are now easily checked. Indeed, in the notation of Theorem 4, we have for squarefree d, $\delta(d) \asymp 1/d$,

$$\#\{p \leq n : \tau(p) \equiv 0 \pmod{d}\} = \delta(d)\pi(n) + O(d^3 n^{1/2} \log nd), \tag{8}$$

assuming GRH. A quasi-GRH allows us to replace $n^{1/2}$ with n^θ with $\theta < 1$. In all cases, this allows us to verify (d) of Theorem 4 for a suitable η. We then find that

$$\frac{\omega(\tau(p)) - \log\log p}{\sqrt{\log\log p}}$$

satisfies an Erdös-Kac law (modulo a quasi-generalized Riemann hypothesis).

(c) *The Liu theorem*: In 2006, Yu–ru Liu [20] proved the following theorem. Let E be a non-CM elliptic curve. For each prime p of good reduction, consider $\omega(|E(\mathbb{F}_p)|)$. Then, subject to a quasi-generalized Riemann hypothesis, $\omega(|E(\mathbb{F}_p)|)$ satisfies an Erdös-Kac law. Again, this is easily derived from our axiomatic framework combined with the theory of ℓ-adic representations and the Chebotarev density theorem. We omit the details since it is now clear how to proceed. In this context, we should mention that the essential ingredients of Liu already appear in the earlier paper of Miri and Murty [22] where it is shown using sieve theory and GRH that $|E(\mathbb{F}_p)|$ has a bounded number of prime factors for infinitely many values of the prime p.

(d) *The Joshi theorem*: Given a newform f, of weight k and level N, with trivial Nebentypus χ_0, Joshi [14] studied $N_p(f) := \chi_0(p)p^{k-1} + 1 - a_f(p)$. He showed that an Erdös-Kac type theorem holds for $\omega(N_p(f))$. Joshi's theorem is now immediate from our axiomatization. If the form is of CM type, then one can dispense with GRH since the analogue of the Bombieri–Vinogradov theorem needed is known unconditionally.

Thus, these result follows immediately from our formalism. One need only verify the axioms and this is easily done in each case using the Chebotarev density theorem and the associated Galois representations.

8 Applications to sums of modular forms

We can now prove Theorem 1. In [27], we showed that if we take two normalised Hecke eigenforms f and g of weight two, with integer coefficients, that are "independent" (in the sense of Loeffler [21]), then the normal number of prime factors of $a_f(p) + a_g(p)$ is $\log\log p$. We can now deduce the analogue of the Erdös-Kac theorem in this context using our formalism. As in [27], we need to remove from our discussion those primes p for which $a_f(p) + a_g(p) = 0$ and this is easily done by an appeal to Theorem 10 of [27]. Modulo a quasi-generalized Riemann hypothesis, the number of such primes is $O(x^\theta)$ for some $\theta < 1$. Thus, we can remove these primes from our discussion. We now define $S(n, d)$ as the number of primes $p \le n$ such that $a_f(p) + a_g(p) \neq 0$ and d divides $a_f(p) + a_g(p)$. We have to verify axioms (a), (b), (c) and (d). To do so, we quote the relevant results from our earlier paper [27].

Axiom (a) requires our sequence to have polynomial growth and this is clear. To check the other axioms, we note that by the work of Loeffler [21], Ribet [34] and Momose [23], the image of the representation

$$\rho_f \times \rho_g : G_{\mathbb{Q}} \to GL_2(\widehat{\mathbb{Z}}) \times GL_2(\widehat{\mathbb{Z}}),$$

is open for f and g independent and of non-CM type. What this means in our context is the following. For any squarefree d, reducing the map $\pmod d$ gives

$$\rho_f \times \rho_g : G(\mathbb{Q}) \to GL_2(\mathbb{Z}/d\mathbb{Z}) \times GL_2(\mathbb{Z}/d\mathbb{Z}),$$

which factors through a finite quotient. Thus, as before, there is a natural number Δ (determined by the ramification of ρ_f and ρ_g) such that if d is squarefree and coprime to Δ, then

$$S(n,d)=\delta(d)\pi(n)+O(d^3x^{1/2}\log(\Delta dx)),\qquad \delta(d)=1/d+O(1/d^2),$$

assuming the generalized Riemann hypothesis for Dedekind zeta functions and the Artin holomorphy conjecture. If we only assume a quasi-generalized Riemann hypothesis, the error term is replaced by $O(d^6x^\theta\log(\Delta dx))$ for some $\theta < 1$. This is sufficient for our purposes and we can immediately verify axioms (b), (c) and (d). The intricate details needed to check the axioms are all contained in our earlier paper [27]. This completes the proof of Theorem 1.

As remarked in [27], one can apply the study of prime divisors of $P(a_f(p), a_g(p))$ where $P(x, y) \in \mathbb{Z}[x, y]$. The formalism goes through without any difficulty. Our discussion was restricted to $P(x, y) = x + y$ for the sake of simplicity of exposition.

9 Localized Erdös-Kac theorems and unconditional results

We discuss here unconditional results with regard to our theorems of the previous sections. We indicate how to prove Theorem 2 and its variations. Our goal is to show that if we "localize" our discussion, then we can derive unconditional results. More precisely, let κ be fixed and positive and let $\omega^*(n)$ denote the number of prime factors of n which are less than $(\log n)^\kappa$. Then, we can derive an Erdös-Kac law for $\omega^*(\tau(p))$ and allied questions for a suitable κ. The general result for $\omega^\dagger(\tau(p))$ follows upon noting that $\omega^\dagger(n) - \omega^*(n)$ is bounded. A similar comment applies to the Erdös-Kac law for $\omega^\dagger(a_f(p) + a_g(p))$.

More precisely, we will show that for any $\kappa < 1/8$,

$$\lim_{n\to\infty} P_n^*\left(p : \frac{\omega^*(\tau(p)) - \log\log\log p}{\sqrt{\log\log\log p}} \le x\right) = \Phi(x), \tag{9}$$

with analogous results for other Fourier coefficients of Hecke eigenforms. Similarly, for any $\kappa < 1/14$,

$$\lim_{n\to\infty} P_n^*\left(p : \frac{\omega^*(a_f(p)+a_g(p)) - \log\log\log p}{\sqrt{\log\log\log p}} \le x\right) = \Phi(x).$$

The unconditional effective Chebotarev density theorem is too weak to give results of earlier sections since its' range of applicability is constrained by various factors. Indeed, if $K/\mathbb{Q}$ is a Galois extension of degree n_K, with Galois group G, and C is a conjugacy class of G, then using notation of Proposition 6, we have the following unconditional theorem of Lagarias and Odlyzko [17]: if $x \ge \exp(10n_K(\log|d_K|)^2)$, then

$$\left|\pi_C(x) - \frac{|C|}{|G|}\pi(x)\right| \le c_1\xi\frac{|C|}{|G|}\pi(x^\beta) + c_2x\exp\left(-c_3\sqrt{\frac{\log x}{n_K}}\right), \tag{10}$$

where c_1, c_2, c_3 are absolute constants, and β is the "exceptional zero" with $\xi = 1$ if it exists and zero otherwise. This exceptional zero (often called a Siegel zero) if it exists, is real and lies in the interval

$$1 - \frac{1}{4\log|d_K|} < \beta < 1.$$

An important theorem of Stark [36] states that

$$\beta < 1 - \frac{c}{|d_K|^{1/n_K}}$$

for some absolute constant c. This leads to two strands of unconditional theorems, depending on whether a Siegel zero exists or not.

We will need an unconditional bound for the number of primes $p \leq n$ for which $\tau(p) = 0$. We can use the unconditional effective version of the Chebotarev density theorem (10) for this purpose. Applying the reasoning of the earlier section to (10) and using the trivial bound $\beta < 1$, we have for any prime ℓ,

$$\#\{p \leq n : \tau(p) \equiv 0 \pmod{\ell}\} \ll \frac{\pi(n)}{\ell} + n \exp\left(-c_3\sqrt{\frac{\log n}{\ell^4}}\right).$$

The argument follows the template of the Murty-Murty theorem discussed in (b) of Sect. 8 above. Choosing $\ell = [(\log n)^{1/12-\epsilon}]$ for any $0 < \epsilon < 1/12$ yields

$$\#\{p \leq n : \tau(p) = 0\} \ll \frac{\pi(n)}{(\log n)^{1/12-\epsilon}}.$$

Better estimates can be derived but they are not needed here. For example, the estimate

$$\#\{p \leq n : \tau(p) = 0\} \ll \frac{\pi(n) \log\log n}{\log n}$$

is derived in [30].

We now apply our theorem to derive an Erdös-Kac law for $\omega^*(\tau(p))$. Let us first suppose that the Siegel zero does not exist. Then, the error term in (10) shows that our axiomatization of an Erdös-Kac type theorem for ω^* requires verification of axiom (d) in particular, which is valid provided $d < (\log n)^\kappa$ with κ suitably small. In the case of the Ramanujan τ-function, or more generally, a Hecke eigenform f with integer coefficients, the Galois extensions that emerge in the study of congruences (mod d) have degrees that grow like d^4 as we indicated in (d) of Sect. 7. Thus, we see that if $d < (\log n)^\kappa$ with $\kappa < 1/12$, axiom (d) can be verified without difficulty. A similar analysis holds for sums of Fourier coefficients of Hecke eigenforms of weight two as discussed in the earlier sections. As noted in Lemma 5 of Sect. 3 in [27], the relevant Galois extensions grow like d^7 and hence the constraint $\kappa < 1/21$ emerges taking into account that $x \geq \exp(10 n_K (\log |d_K|)^2)$ for the application of the unconditional Chebotarev density theorem. If on the other hand, there is a Siegel zero, we must study the x^β term in (10) and see how it affects axiom (d). To this end, we apply the estimate (19) on page 134 of Serre [35]:

$$\frac{\log |d_K|}{n_K} \leq \log n_K + \sum_{p \in P(K/\mathbb{Q})} \log p,$$

where $P(K/\mathbb{Q})$ is the set of primes that ramify in K. This means that

$$|d_K|^{1/n_K} \leq n_K \prod_{p \in P(L/K)} p.$$

As was seen in the previous sections, the primes that ramify in the Galois extensions arising from (mod d) reduction are (apart from primes dividing the level of the modular forms) prime divisors of d. Thus, a crude bound of

$$|d_K|^{1/n_K} \ll n_K^2,$$

suffices for our purposes and leads to $\kappa < 1/8$ in the case of the Ramanujan τ-function and $\kappa < 1/14$ in the case of the sum of two Hecke eigenforms of weight two with integer coefficients. The discussion of these results related to the cognate function $\Omega^*(n)$ which counts the number of prime factors of n less than $(\log n)^\kappa$ with multiplicity can be derived following the template of Sect. 5. In all cases, this shows that for some $\kappa > 0$, we can apply Theorem 5.

10 Concluding remarks

The study of localized Erdös-Kac theorems in the classical context was initiated in [5]. There, a more general theorem with the study of prime factors in larger ranges was discussed. However, in the type of examples discussed in this paper, especially in the context of application of the Chebotarev density theorem, we are constrained in the range of applicability. Still, the unconditional results do give a normal distribution as suggested by the conditional results. It will be interesting to see to what extent these methods can be pushed. For instance, the dichotomy evident in the last section, giving two types of results dependent on the existence or non-existence of the Siegel zero is one case in point, where perhaps some strengthening can ensue. It may be possible to enlarge the range of κ by using the upper bounds derived for $\pi_C(x)$ in [18]. A local version of the Murty–Saidak theorem is possible. Here, the appearance of the Siegel zero term can be eliminated and one can derive a result valid for $\kappa < 1/2$. A similar remark can be made for the result of Liu [20]. The range of $\omega^\dagger(n)$ can be extended a bit. For instance, since we have established a normal law for $\omega^\dagger(\tau(p))$, the same law is valid for the enumeration of prime divisors less than $(\log n)^{f(n)}$ with $f(n) = (\log\log\log n)^\alpha$ for any $\alpha < 1/2$. Similar remarks apply to the other sequences in our litany of applications. We should point out that the study of the normal order of general multiplicative functions, in the settings we have described here, begins in the work of the first two authors [25], [26] in the context of modular forms. As we noted there, it can be also applied to the Euler φ function as in [8]. In the latter context, Tenenbaum [37] obtained finer results with error terms (see Corollary 2.6 of [37]). Similar results can be derived in the modular context as well. Finally, we remark that one can extend these results to sums of Fourier coefficients of three or more Hecke eigenforms in view of results obtained in [21]. Though theorems from Galois representation theory are now in place, the technical apparatus to study the general situation is still in evolution. In addition, condition (d) of Sect. 3 is not the usual estimate of Bombieri–Vingoradov type that one often encounters in sieve theory as we need only a saving of an arbitrary power of $\log\log n$ instead of $\log n$, So, this may be cause for optimisim that some of our conditional results can be made uncoditional. This will be taken up in future work.

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