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Transcendental values of certain Eichler integrals

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Abstract

We study the transcendence of certain Eichler integrals associated to Eisenstein series and more generally to modular forms using functional identities due to Ramanujan, Grosswald, Weil *et al.* The special values of such integrals at algebraic points in the upper half-plane are linked to Riemann zeta values at odd positive integers.

1. *Introduction*

One of the most remarkable formulas suggested by Ramanujan is the following identity [**28**, vol II, p. 171, Corollary iv; also **4**, p. 276] involving the odd values of the Riemann zeta function:

$$\alpha^{-k}\left\{\frac{1}{2}\zeta(2k+1)+\sum_{n=1}^{\infty}\frac{n^{-2k-1}}{e^{2\alpha n}-1}\right\}=(-\beta)^{-k}\left\{\frac{1}{2}\zeta(2k+1)+\sum_{n=1}^{\infty}\frac{n^{-2k-1}}{e^{2\beta n}-1}\right\}$$
$$-2^{2k}\sum_{j=0}^{k+1}(-1)^j\frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!}\alpha^{k+1-j}\beta^j, \qquad (1)$$

where $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$, k is any non-zero integer and B_j is the jth Bernoulli number.

For a fascinating account of the history of this formula, we refer the reader to [**3**]. Apart from its intrinsic beauty, several interesting corollaries can be derived from this formula (1) of Ramanujan. Here are some.

First, if we put $\alpha = \beta = \pi$ and k is even, then we deduce

$$\sum_{j=0}^{k+1}(-1)^j\frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!}=0. \qquad (2)$$

Second, if we put $\alpha = \beta = \pi$ and k is odd, then we deduce

$$\zeta(2k+1)+2\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2\pi n}-1)}=\pi^{2k+1}2^{2k}\sum_{j=0}^{k+1}(-1)^{j+1}\frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!}, \qquad (3)$$

a formula apparently due to Lerch [**23**] and published in an obscure journal (see also [**3**]). Since the left-hand side of this equation is non-zero (both the terms being positive), the right-hand side is a non-zero rational multiple of π^{2k+1}. Consequently, at least one of

$$\zeta(4k+3), \qquad \sum_{n=1}^{\infty}\frac{1}{n^{4k+3}(e^{2\pi n}-1)}$$

is transcendental for every integer $k \geqslant 0$. For example, we have the elegant formula

$$\zeta(3)+2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)}=\frac{7\pi^3}{180}.$$

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Using the theory of the Epstein zeta function, Terras [35] found the related formula

$$\zeta(3) = \frac{2\pi^3}{45} - 4\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-3}(n)\left(2\pi^2 n^2 + \pi n + \frac{1}{2}\right),$$

where

$$\sigma_t(n) = \sum_{d|n} d^t.$$

These formulas suggest that there is some interest in studying (Lambert type) series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^k(e^{2\pi nz}-1)}.$$

By developing the denominator into a power series and after a suitable rearrangement, we are led to define, following Grosswald [16],

$$F_k(z) = \sum_{n=1}^{\infty} \sigma_{-k}(n)\, e^{2\pi inz}.$$

We may also write

$$F_k(z) = \sum_{m,n=1}^{\infty} n^{-k}\, e^{2\pi imnz} = \sum_{n=1}^{\infty} \frac{1}{n^k}\left(\frac{e^{2\pi inz}}{1-e^{2\pi inz}}\right) = -\zeta(k) - \sum_{n=1}^{\infty} \frac{1}{n^k(e^{2\pi inz}-1)}.$$

For odd negative values of $k \leqslant -3$, $F_k(z)$ is essentially the Eisenstein series $E_{1-k}(z)$ (minus the constant term) of weight $1-k$ for the full modular group. For odd positive values of k, one can view $F_k(z)$ as an Eichler integral (in the sense of Goldstein [12] and Razar [32]). Indeed, since

$$\sigma_{-k}(n) = n^{-k}\sigma_k(n),$$

we may rewrite $F_k(z)$ as

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^k}\, e^{2\pi inz},$$

and it is now clear that $F_k(z)$ can be obtained by successive integration of the classical Eisenstein series minus its constant term. More precisely, for even $k > 2$, let E_k be defined as

$$E_k(z) = \gamma_k + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, e^{2\pi inz}, \quad \gamma_k = \frac{-B_k}{2k}.$$

Then, for any odd $k > 1$, we have

$$F_k(z) = \frac{(2\pi i)^k}{(k-1)!}\int_{i\infty}^{z} [E_{k+1}(\tau) - \gamma_{k+1}](\tau - z)^{k-1}\, d\tau,$$

which is an Eichler integral of the first kind. Much research has been done on the transcendental values of the Eisenstein series [26, 27], but very little is known about the transcendence of Eichler integrals. These integrals are intricately linked to zeta values, instances of which can be traced to Beukers's proof [6] of irrationality of $\zeta(3)$.

As discussed in the next section, a recent result in [25] implies that, for a given $k \geqslant 4$, there exist algebraic numbers α in the upper half-plane $\mathfrak{H}$ depending on k such that the numbers

$$F_{2k+1}(\alpha) - \alpha^{2k} F_{2k+1}(-1/\alpha)$$

are non-zero algebraic multiples of $\zeta(2k+1)$. This motivates the investigation of the values of the function $F_{2k+1}(z) - z^{2k}F_{2k+1}(-1/z)$ at algebraic points in the upper half-plane. We have the following result.

THEOREM 1.1. *Let k be a non-negative integer and set $\delta = 0, 1, 2, 3$ accordingly as the* $\gcd(k, 6)$ *equals* $1, 2, 3$ *or* 6, *respectively. With at most $2k + 2 + \delta$ exceptions, the number*

$$F_{2k+1}(\alpha) - \alpha^{2k} F_{2k+1}(-1/\alpha)$$

is transcendental for every algebraic $\alpha \in \mathfrak{H}$. In particular, there are at most $2k + 2 + \delta$ algebraic numbers $\alpha \in \mathfrak{H}$ such that $F_{2k+1}(\alpha)$ and $F_{2k+1}(-1/\alpha)$ are both algebraic.

The results in [**25**] show that the numbers i, ρ and $-\rho^2$ are examples of such exceptional algebraic numbers. Here $\rho = e^{2\pi i/3}$ is the cube root of unity in the upper half-plane. However, $F_{2k+1}(\alpha) - \alpha^{2k} F_{2k+1}(-1/\alpha)$ is zero at each of these points. It is highly improbable that any other exceptional algebraic number exists. Indeed, if such an α exists, then one can show that $\zeta(2k + 1)$ is an algebraic linear combination of 1 and π^{2k+1}, which is unlikely. In this context, there is a weaker conjecture due to Kohnen [**21**], which predicts that the quotient $\zeta(2k + 1)/\pi^{2k+1}$ is transcendental for all odd integers $k \geqslant 1$. The irrationality of these quotients and, more generally, the irrationality of $\zeta(2k + 1)/\pi^{2k+1}$ for all $k \geqslant 1$ will also follow from a conjecture of Chowla and Milnor (see [**19**]).

Indeed, Kohnen [**21**] has made general conjectures regarding special values of L-series attached to modular forms of weight $2k$ for the full modular group. These conjectures imply the transcendence of $\zeta(2k + 1)/\pi^{2k+1}$ for all odd integers $k \geqslant 1$. To be precise, Kohnen defines two vector spaces as follows. Let K be a number field and let $M_{2k}(K)$ be the space of modular forms of weight $2k$ whose Fourier coefficients lie in K. Let $M^{\pm}_{2k}(K)$ be the space of modular forms f satisfying $L(j + 1, f)/\pi^{j+1} \in K$ for all $0 \leqslant j \leqslant 2k - 2$, $(-1)^j = \pm 1$. Here, $L(s, f)$ is the L-series attached to f. The spaces $S_{2k}(K)$ and $S^{\pm}_{2k}(K)$ are analogously defined for cusp forms. Kohnen's conjecture is that, for any algebraic number field K, we have $M^{+}_{2k}(K) \cap M_{2k}(K) = \{0\}$ and $S^{-}_{2k}(K) \cap S_{2k}(K) = \{0\}$. This along with a result of Manin [**24**] implies that, for any normalized Hecke eigen cuspform f, $L(j + 1, f)/\pi^{j+1}$ is either zero or transcendental for all $0 \leqslant j \leqslant 2k - 2$. Applying Kohnen's conjecture to the classical Eisenstein series, we have the transcendence of $\zeta(2k + 1)/\pi^{2k+1}$ for all odd integers $k \geqslant 1$.

The structure of our paper is as follows. In the next section, we introduce a generalization of Ramanujan's identity by Grosswald and discuss the recent work of Murty, Smyth and Wang [**25**], which motivates our Theorems 1.1 and 2.1. Section 3 gives the proof of Theorem 1.1 mentioned in the introduction. In the next section, we introduce a rather sweeping generalization of Ramanujan's identity by Grosswald. Part of our goal also is to bring out some of the positive features of Grosswald's generalization. For instance, in the final section we use this in a different context to locate the zeros of certain cusp forms. The generalization of Grosswald has been rediscovered in the modular context by a variety of authors, the most notable being Weil [**36**]. These modular analogues of Ramanujan's identity are investigated in Section 5. The difference of corresponding Eichler integrals in the modular setting is related to an algebraic linear combination of special values of the form $L(j, f)/\pi^j$. Here $L(s, f)$ is the L-function of the associated modular form f and j varies over certain integers in the critical strip. In the special case of weight 2 modular forms, this difference is an algebraic multiple of $L(1, f)/\pi$. In the penultimate section, we restrict our attention to this special case relating it to the works of Bertrand and Chudnovsky.

2. *Ramanujan's formula revisited*

Motivated perhaps by Ramanujan's identity, Grosswald, in a series of papers [**14**–**17**], investigated various convergent series representations for $\zeta(2k + 1)$. In fact, in [**14**], he derived the following generalization of (1): for any z lying in the upper half-plane $\mathfrak{H}$,

$$\begin{aligned} F_{2k+1}(z) - z^{2k} F_{2k+1}(-1/z) = \frac{1}{2}\zeta(2k+1)(z^{2k}-1) \\ + \frac{(2\pi i)^{2k+1}}{2z} \sum_{j=0}^{k+1} z^{2k+2-2j} \frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!}. \end{aligned} \tag{4}$$

Putting $z = i\beta/\pi$, with $\beta > 0$, gives Ramanujan's formula (1). As noted in later sections, this formula is a special case of a general formalism. For the purpose of our present discussion, it is convenient to introduce the *Ramanujan polynomial*, which appears on the right-hand side of the above formula:

$$R_{2k+1}(z) = \sum_{j=0}^{k+1} z^{2k+2-2j} \frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!}.$$

It is clear from Grosswald's formula that zeros of this polynomial lying in the upper half-plane that are not $2k$th roots of unity, if any, will give rise to explicit formulas for $\zeta(2k+1)$ in terms of special values of the Eichler integrals $F_{2k+1}(z)$.

In a recent paper, Murty, Smyth and Wang [**25**] have shown that, for $k \geqslant 4$, this is indeed the case. More precisely, they showed that, for $k \geqslant 4$, all the zeros of $R_{2k+1}(z)$ are simple and lie on the unit circle apart from four real roots. Moreover, the only possible roots of unity which are zeros of $R_{2k+1}(z)$ are $\pm i$ (which happens if and only if k is even) and $\pm\rho, \pm\rho^2$ with $\rho = e^{2\pi i/3}$ (which happens if and only if 3 divides k). As a consequence, they deduce that, for $k \geqslant 4$, there is an algebraic α with $|\alpha| = 1$ which is not a $2k$th root of unity so that

$$\zeta(2k+1) = \frac{2}{\alpha^{2k}-1}[F_{2k+1}(\alpha) - \alpha^{2k}F_{2k+1}(-1/\alpha)];$$

that is, $\zeta(2k+1)$ can be written as an algebraic linear combination of two Eichler integrals. Thus, it is of interest to consider the function

$$G_{2k+1}(z) := \frac{1}{z^{2k}-1}(F_{2k+1}(z) - z^{2k}F_{2k+1}(-1/z)),$$

and study its special values.

For instance, any zero $z \in \mathfrak{H}$ of $G_{2k+1}(z)$ with $z^{2k} \neq 1$ is expected to be transcendental and not lying in the field $\bar{\mathbb{Q}}(\pi)$. Indeed, if z_0 is a zero, then the Ramanujan–Grosswald formula implies that $\zeta(2k+1) \in \bar{\mathbb{Q}}(z_0, \pi)$.

We now have the following theorem.

THEOREM 2.1. *The set*

$$\{G_{2k+1}(z) : z \in \mathfrak{H} \cap \bar{\mathbb{Q}}, z^{2k} \neq 1\}$$

contains at most one algebraic number.

Proof. Indeed, suppose that there are distinct algebraic numbers α_1, α_2 lying in the upper half-plane such that $G_{2k+1}(\alpha_1)$ and $G_{2k+1}(\alpha_2)$ are both algebraic and distinct. By the Ramanujan–Grosswald formula (4) we have, for $i = 1, 2$, that

$$G_{2k+1}(\alpha_i) = \tfrac{1}{2}\zeta(2k+1) + R^*_{2k+1}(\alpha_i)\pi^{2k+1},$$

where $2z(z^{2k}-1)R^*_{2k+1}(z) = (2i)^{2k+1}R_{2k+1}(z)$. By subtracting these two relations, we obtain

$$\pi^{2k+1}(R^*_{2k+1}(\alpha_1) - R^*_{2k+1}(\alpha_2)) = G_{2k+1}(\alpha_1) - G_{2k+1}(\alpha_2) \in \bar{\mathbb{Q}}.$$

Since $G_{2k+1}(\alpha_1) \neq G_{2k+1}(\alpha_2)$, the right-hand side is non-zero and, consequently, the left-hand side also is. This implies that π^{2k+1} is algebraic, which is a contradiction. $\square$

Presumably, for reasons alluded to above, there are no algebraic numbers in the set. For otherwise, $\zeta(2k+1)$ would be an algebraic linear combination of 1 and π^{2k+1}, a highly unlikely possibility.

The study of formula (4) leads to some new expressions for $\zeta(2k+1)$. We already highlighted Lerch's formula (3), which gives a formula for $\zeta(4k+3)$. If $\rho = e^{2\pi i/3}$, then $-1/\rho = \rho + 1$, so that $F_{2k+1}(-1/\rho) = F_{2k+1}(\rho+1) = F_{2k+1}(\rho)$. This observation leads to the following seemingly new formula for $\zeta(2k+1)$ for $(k,3) = 1$, namely,

$$\zeta(2k+1) + 2F_{2k+1}(\rho) = \frac{(2\pi i)^{2k+1}}{\rho(1-\rho^{2k})} R_{2k+1}(\rho). \tag{5}$$

In other words, if 3 does not divide k, then

$$\zeta(2k+1) + 2\sum_{m=1}^{\infty} \frac{1}{m^{2k+1}((-1)^m e^{\pi m\sqrt{3}} - 1)} \in \pi^{2k+1}\bar{\mathbb{Q}}^*.$$

This can be regarded as the counterpart of Lerch's formula (3). If k is even, then both sides of Grosswald's formula (4) vanish for $z = i$ by the results of [**25**]. However, we can factor out the root and take limits. Indeed, if $2 \mid k$, then the results of [**25**] show that both the numerator and denominator of $G_{2k+1}(z)$ vanish at $z = i$. Using L'Hôpital's rule, we obtain

$$\zeta(2k+1) + 2\sum_{n=1}^{\infty} \frac{1}{n^{2k+1}(e^{2\pi n} - 1)} + \frac{4\pi}{k}\sum_{n=1}^{\infty} \frac{e^{2\pi n}}{n^{2k}(e^{2\pi n} - 1)^2} \in \pi^{2k+1}\bar{\mathbb{Q}}^*.$$

If $3 \mid k$, by the results of [**25**] both sides of Grosswald's formula (4) vanish for $z = \rho$ and, following the above described recipe, then we get

$$k\rho^2\zeta(2k+1) + (\rho - 1)F'_{2k+1}(\rho) + 2k\rho^2 F_{2k+1}(\rho) \in \pi^{2k+1}\bar{\mathbb{Q}}^*.$$

3. *Proof of Theorem* 1.1

We begin by considering algebraic numbers $\alpha \in \mathfrak{H}$ that are not $2k$th roots of unity. By Theorem 2.1, for all such α, the function

$$F_{2k+1}(z) - z^{2k}F_{2k+1}(-1/z)$$

can take at most one algebraic value, say A. Thus, all such α are necessarily roots of the following polynomial:

$$\frac{1}{2}\zeta(2k+1)(z^{2k} - 1)z + \frac{(2\pi i)^{2k+1}}{2}\sum_{j=0}^{k+1} z^{2k+2-2j}\frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!} = Az.$$

Thus, there are at most $2k+2$ algebraic numbers in the upper half-plane that are not $2k$th roots of unity and for which $F_{2k+1}(\alpha) - z^{2k}F_{2k+1}(-1/\alpha)$ is algebraic.

Now we consider algebraic numbers $\alpha \in \mathfrak{H}$ that are $2k$th roots of unity. For such an α, formula (4) implies that

$$F_{2k+1}(\alpha) - \alpha^{2k}F_{2k+1}(-1/\alpha) = \frac{(2\pi i)^{2k+1}}{2\alpha}R_{2k+1}(\alpha).$$

If, for such an α, $F_{2k+1}(\alpha) - z^{2k}F_{2k+1}(-1/\alpha)$ is algebraic, then transcendence of π implies that $R_{2k+1}(\alpha) = 0$. Then, by the theorem of [**25**] cited earlier, this cannot happen if $\gcd(k,6) = 1$. If $\gcd(k,6)$ is equal to 2 or 3, then $\alpha = i$ and $\alpha = e^{2\pi i/3}, -e^{-2\pi i/3}$ are the only possible values in each of the respective cases. In both of these cases, we have

$$F_{2k+1}(\alpha) - \alpha^{2k} F_{2k+1}(-1/\alpha) = 0.$$

Finally, if $(k, 6) = 6$, then all three values are zeros of the function in question. This completes the proof. □

The above proof as well as the remark following the proof of Theorem 2.1 suggests the following conjecture.

CONJECTURE. Any zero of the function

$$F_{2k+1}(z) - z^{2k} F_{2k+1}(-1/z)$$

in the upper half-plane other than $i, e^{2\pi i/3}$ and $-e^{-2\pi i/3}$ is transcendental.

We end the section with some explicit examples. In the case of $k = 1$, we have

$$F_3(z) - z^2 F_3(-1/z) = \frac{1}{2}\zeta(3)(z^2 - 1) + \frac{(2\pi i)^3}{2z} R_3(z),$$

where

$$R_3(z) = -\frac{z^4}{6!} + z^2 \frac{1}{144} - \frac{1}{6!}.$$

There are no roots of this polynomial lying in the upper half-plane. More generally, it has been shown in [25] that $R_{2k+1}(z)$ has four distinct real zeros for all $k \geqslant 1$.

To cite another example, in the case of $k = 3$, the corresponding Ramanujan polynomial is given by

$$R_7(z) = \frac{-3z^8 + 10z^6 + 7z^4 + 10z^2 - 3}{10!}.$$

This has exactly two zeros on the upper half-plane, namely, ρ and $-\bar{\rho}$. For $k \geqslant 4$, as shown by Murty, Smyth and Wang [25], $\zeta(2k + 1)$ is expressible as an algebraic multiple of the difference of two Eichler integrals. Further numerical data can be found in their paper.

4. *Grosswald's lemma*

The derivation of Grosswald, extending Ramanujan's formula, is a special case of a galaxy of identities arising from Dirichlet series satisfying functional equations. Bochner [7] seems to have been the first to state such identities, although his paper indicates that he was unaware of Ramanujan's formula. Later, Chandrasekharan and Narasimhan [8] extended these thoughts further, still unaware of the earlier work by Ramanujan. Finally, it has been again rediscovered as late as 1976, by Weil [36] in the modular setting. This version has been discussed by various authors, notably Goldstein [12] and Razar [32]. Investigation of a similar nature has also been carried out by Apostol [1].

The most comprehensive generalization occurs in a paper of Grosswald [17], who indicated a general method that includes the result generally attributed to Weil [36]. This appears as 'Main Lemma' in p. 116 of Grosswald's paper. The extreme generality of the result makes it appear unwieldy and conceals some of the more elegant results that can be derived from it. This partly explains how it has been rediscovered in the modular context by a variety of authors, the most notable being Weil [36].

One of our purposes is to bring out some of the positive features of Grosswald's lemma to the foreground. For instance, equation (4), which includes Ramanujan's formula (1), is a consequence of this general formula.

Following [17], for $s \in \mathbb{C}$ with real part σ, let

$$\Delta(s) = \prod_{\nu=1}^{M} \Gamma(\alpha_\nu s + \beta_\nu), \quad \alpha_\nu > 0,\ \beta_\nu \in \mathbb{C}.$$

Suppose a Dirichlet series

$$\phi(s) = \sum_{n=1}^{\infty} a(n)\, e^{-\lambda_n s}$$

is convergent for $\sigma \geqslant \sigma_0 > 0$ with very mild growth conditions (see [17, p. 115]) and define, for $s \in \mathbb{C}$,

$$\Phi(s) = \phi(s)\Delta(s)P(s),$$

where $P(s)$ is a rational function. Suppose that

$$\Phi(s) = (-1)^{\delta}\Phi(r-s)$$

with $\delta = 0$ or $\delta = 1$ and real r. For $z \in \mathfrak{H}$, let

$$F(z) = \frac{1}{2\pi i}\int_{(\sigma_2)} \Phi(s)(z/i)^{-s}\, ds,$$

where $\sigma_2 = \sigma_0 + \epsilon$ for ϵ sufficiently small and $\int_{(\sigma)} = \lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT}$. Also define, for $u \in \mathbb{C}$ and $\sigma_1 = r - \sigma_2$ (with suitable σ_0),

$$S(u) = \sum_{\sigma_1 \leqslant \sigma \leqslant \sigma_2} \mathrm{Res}\{\Phi(s)u^s\}.$$

Then Grosswald [17, p. 116] proved that, for $z \in \mathfrak{H}$,

$$F(-1/z) - (-1)^{\delta}(z/i)^{r}F(z) = S(z/i),$$

which, in particular, shows that

$$2iF'(i) + rF(i) = -\sum_{\sigma_1 \leqslant \sigma \leqslant \sigma_2} \mathrm{Res}\{s\Phi(s)\} \quad \text{if } \delta = 0, \tag{6}$$

$$2F(i) = \sum_{\sigma_1 \leqslant \sigma \leqslant \sigma_2} \mathrm{Res}\{\Phi(s)\} \quad \text{if } \delta = 1. \tag{7}$$

If we apply this lemma to the series $\phi(s) = \zeta(s)\zeta(s+2k+1)$, which satisfies the functional equation

$$\Phi(s) = (-1)^{k}\Phi(-2k-s),$$

with

$$\Phi(s) = (2\pi)^{-s}\Gamma(s)\phi(s),$$

we deduce Ramanujan's formula, or more precisely (4) of which Ramanujan's formula is a special case.

5. *The modular case*

As noted earlier, the modular analogue of Ramanujan's formula had been worked out by many authors, the most notable being Weil [36]. Razar [32] and Weil [36] derive the following result (see [32, Theorem 2]). Let

$$f(z) = \sum_{n=0}^{\infty} a(n)\, e^{2\pi i n z/\lambda},$$

$$g(z) = \sum_{n=0}^{\infty} b(n)\, e^{2\pi i n z/\lambda},$$

$$f^*(z) = \frac{a(0)z^{k-1}}{(k-1)!} + \left(\frac{2\pi i}{\lambda}\right)^{-(k-1)} \sum_{n=1}^{\infty} \frac{a(n)}{n^{k-1}} e^{2\pi i n z/\lambda},$$

$$g^*(z) = \frac{b(0)z^{k-1}}{(k-1)!} + \left(\frac{2\pi i}{\lambda}\right)^{-(k-1)} \sum_{n=1}^{\infty} \frac{b(n)}{n^{k-1}} e^{2\pi i n z/\lambda}.$$

The functions f^*, g^* are examples of standard Eichler integrals of the second kind. Let

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

be the associated L-function for f. If k is a positive integer and γ a complex number such that

$$f(z) = \gamma z^{-k} g(-1/z),$$

then the result of Razar–Weil states that

$$f^*(z) - \gamma z^{k-2} g^*(-1/z) = \sum_{j=0}^{k-2} \frac{L(k-1-j, f)}{j!} \left(\frac{2\pi i}{\lambda}\right)^{-(k-1-j)} z^j. \tag{8}$$

In most applications, λ and γ would be algebraic numbers and we shall assume this henceforth. The numbers $L(n, f)/\pi^n$ are the natural generalizations of periods of modular forms as introduced by Kohnen in relation to his conjecture. Here we have the following theorem.

THEOREM 5.1. *Let f, g be as above with γ and λ algebraic numbers. Suppose that at least one of $L(n, f)/\pi^n$ for $1 \leqslant n \leqslant k-1$ is transcendental. Then, there are at most $k-2$ algebraic values of z in the upper half-plane such that*

$$f^*(z) - \gamma z^{k-2} g^*(-1/z)$$

is algebraic.

Proof. Let $\alpha_1, \ldots, \alpha_{k-1}$ be distinct algebraic numbers in $\mathfrak{H}$ such that

$$\beta_i := f^*(\alpha_i) - \gamma \alpha_i^{k-2} g^*(-1/\alpha_i)$$

is algebraic for $1 \leqslant i \leqslant k-1$.

From (8), we deduce

$$(L(k-1, f)\lambda^{k-1}/(2\pi i)^{k-1}, \ldots, L(1, f)\lambda/[(k-2)!(2\pi i)])V = (\beta_1, \ldots, \beta_{k-1}),$$

where V is the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{k-2} & \alpha_2^{k-2} & \cdots & \alpha_{k-1}^{k-2} \end{pmatrix}.$$

This matrix is invertible since $\alpha_1, \cdots, \alpha_{k-1}$ are all distinct and, thus, we deduce that all of the numbers $L(n, f)\lambda^n/\pi^n$ are algebraic, which is a contradiction to our hypothesis. $\square$

We end the section with the following remarks.

REMARK 5.1. Let f be a normalized Hecke eigenform of weight k and level N. Works of Eichler, Shimura and Manin [**24**] imply that there are two 'periods', ω_+ and ω_-, such that $L(n, f)/\pi^n$ belongs to $\omega_+\bar{\mathbb{Q}}$ or $\omega_-\bar{\mathbb{Q}}$ according as n is even or odd for $1 \leqslant n \leqslant k-1$. It is perhaps true that the numbers π, ω_+, ω_- are algebraically independent.

REMARK 5.2. As suggested by Goldstein [**12**] and Razar [**32**], formulas of the type (8) are special cases of a more general phenomenon. For Γ equal to $\mathrm{SL}_2(\mathbb{Z})$ or more generally $\Gamma_0(N)$, Γ acts on the space of polynomials $p(z)$ in $\mathbb{C}[z]$ of degree at most k as follows:

$$p \longrightarrow p|_\sigma, \quad \sigma \in \Gamma$$
$$p|_\sigma(z) = p(\sigma z)(cz+d)^k.$$

Then, for $f \in M_{k+2}(\Gamma)$, we can consider the *generalized Eichler integral*

$$F(z, p) = \int_{z_0}^{z} f(\tau)p(\tau)\,d\tau, \quad z_0 \in \mathfrak{H}.$$

Special cases of such a formalism are the following:

$$F_0(z) = \frac{1}{k!}\int_{z_0}^{z} f(\tau)(\tau - z)^k\,d\tau,$$
$$G_0(z) = \frac{a(0)z^{k+1}}{(k+1)!} + \left(\frac{2\pi i}{\lambda}\right)^{-(k+1)} \sum_{n=1}^{\infty} \frac{a(n)}{n^{k+1}} e^{2\pi i n z/\lambda}.$$

We have already encountered examples of both types before. These are referred to as the *standard Eichler integrals of the first and the second kind*, respectively. The Eichler integrals were introduced by Eichler for the purpose of constructing what is now known as Eichler–Shimura cohomology [**10**, **34**].

Let F be either of the above special type of Eichler integrals. For any element $\sigma \in \Gamma$, it satisfies the following transformation law:

$$F(\sigma z)(cz+d)^k = F(z) + p_\sigma(z) \quad \sigma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}.$$

Here, as before, $p_\sigma(z)$ is a polynomial of degree at most k and is referred to as the *period* of F with respect to σ.

Goldstein and Razar [**13**] proved the following transformation law for an arbitrary generalized Eichler integral $F(z, p)$. Let $\sigma \in \Gamma$ and p_σ^0 be the period polynomial for the standard Eichler integral $F_0(z)$. Further, let

$$p_\sigma^0(z) = \sum_{j=0}^{k} \alpha_j z^j \quad \text{and} \quad p(z) = \sum_{j=0}^{k} \beta_j z^j.$$

Then

$$F(\sigma z, p) = F(z, p|_\sigma) - k! \sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j}^{-1} \alpha_j \beta_{k-j}.$$

In this set-up, Razar [**32**, Theorem 3] has a generalized version of the identity given by (8) for an arbitrary $\sigma \in \Gamma$. Thus, we can have an appropriate generalization of Theorem 5.1 regarding algebraic values of

$$f^*(z) - (f|\sigma)^*(z),$$

for z in the upper half-plane, provided that we make some mild assumptions about the transcendence of at least one of the coefficients of the period polynomial.

6. *The case $k = 2$*

In the case $k = 2$, the right-hand side of (8) has only one term and we need only to consider the number $L(1, f)/\pi$. More explicitly, denoting by W_N the Atkin–Lehner involution, we have $f(W_N z) = \pm N z^2 f(z)$. If we set

$$F(z) = \sum_{n=1}^{\infty} a(n)\, e^{2\pi i n z/\sqrt{N}},$$

then it is easily seen that

$$F(-1/z) = \pm z^2 F(z).$$

A quick calculation shows that

$$F^*(z) \pm F^*(-1/z) = \sqrt{N} L(1, f)/2\pi i. \tag{9}$$

First, we have the following theorem of Bertrand [5].

Proposition 6.1. *Let $f(z)$ be a normalized Hecke eigenform of weight 2 on $\Gamma_0(N)$. Let τ be a rational number or an element of the upper half-plane such that the modular invariant $j(\tau)$ is algebraic. Then, any determination of the integral*

$$2\pi i \int_{i\infty}^{\tau} f(z)\, dz$$

is either zero or transcendental.

If

$$f(z) = \sum_{n=1}^{\infty} a(n)\, e^{2\pi i n z}$$

is a normalized Hecke cuspidal eigenform of weight 2 on $\Gamma_0(N)$, then Bertrand's theorem implies that the (conditionally convergent) sum

$$\sum_{n=1} \frac{a(n)}{n}$$

is either zero or transcendental.

If $L(1, f) \neq 0$, in the context of identity (8), then we are interested in the nature of $L(1, f)/\pi$.

In the important case when f has integer coefficients, by the works of Eichler and Shimura, f corresponds to an elliptic curve E defined over rationals. A classical result of Schneider implies that any non-zero period of such an elliptic curve is transcendental. In the special case when E has complex multiplication, by the works of Chudnovsky, π and any non-zero period of E are algebraically independent. Consequently, if $L(1, f) \neq 0$, then $L(1, f)/\pi$ is transcendental.

7. *Grosswald's lemma and zeros of modular forms*

In this section, we use Grosswald's lemma to locate zeros of modular forms. For a modular form f of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$, it is easy to see that f has a zero at i if k is odd. Locating zeros of modular forms of higher level is not so straightforward.

Let $f \in S_k(\Gamma_1(N))$ be a cusp form of weight k for $\Gamma_1(N)$ and W_N be the idempotent operator from $S_k(\Gamma_1(N))$ to $S_k(\Gamma_1(N))$, defined by

$$(W_N f)(z) = i^k N^{-k/2} z^{-k} f(-1/Nz).$$

Let

$$S_k^{\pm}(\Gamma_1(N)) = \{f \in S_k(\Gamma_1(N)) \mid W_N f = \pm f\}$$

be the eigenspaces of W_N. Then the completed L-function

$$\Phi(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(s,f)$$

has an analytic continuation to the entire complex plane and satisfies the functional equation

$$\Phi(s) = \pm\Phi(k-s).$$

The sign in the above equation is 1 if f is in $S_k^+(\Gamma_1(N))$ and -1 if f is in $S_k^-(\Gamma_1(N))$. We refer to [**9**, p. 204] for further details. We have the following theorem.

THEOREM 7.1. (1) *If $f \in S_k^-(\Gamma_1(N))$, then f has a zero at $i/\sqrt{N}$.*

(2) *If $f \in S_k^+(\Gamma_1(N))$, then $f' - (ik\sqrt{N}/2)f$ has a zero at $i/\sqrt{N}$. In particular, if f has a zero at $i/\sqrt{N}$, then it is a zero of order at least 2.*

As an example, the new form for $\Gamma_0(4)$

$$f(z) := \eta^{12}(2z) = q - 12q^3 + 54q^5 - 88q^7 - 99q^9 + \ldots$$

does not vanish at $i/2$ and hence is necessarily in the $+1$ eigenspace.

In this context, we refer to the papers [**2**, **11**, **18**, **22**, **29**–**31**, **33**], where zeros of modular forms have been studied. We note that the first part of the theorem can also be deduced from the action of the Atkin–Lehner operator on f. We, however, derive it as an application of Grosswald's lemma.

Proof. Let $f \in S_k^-(\Gamma_1(N))$ and also let

$$f(z) = \sum_{n \geqslant 1} a(n) q^n, \quad q = \exp(2\pi i z),\ z \in \mathfrak{H}$$

be the Fourier expansion of f at infinity. The Dirichlet series

$$\phi(s) = \sum_{n \geqslant 1} a(n)\, e^{-\lambda_n s}, \quad \lambda_n = \log\left(\frac{2n\pi}{\sqrt{N}}\right)$$

is absolutely convergent for $\Re(s) = \sigma > k/2 + 1$. Let us set $\Delta(s) = \Gamma(s)$ and $P(s) \equiv 1$ in the set-up considered by Grosswald (see Section 4). Since $\Phi(s) = -\Phi(k-s)$, we have $\delta = 1$ and $r = k$ in this context. Since the associated $\Phi(s)$ has an analytic continuation to the entire complex plane, we have

$$\sum_{\sigma_1 \leqslant \sigma \leqslant \sigma_2} \operatorname{Res}\{\Phi(s)\} = 0.$$

Here $\sigma_2 = k/2+1+\epsilon$ and $\sigma_1 = k/2-1-\epsilon$. Thus, by (7) in Section 4, we have $F(i)=0$. However

$$\begin{aligned} F(i) &= \frac{1}{2\pi i}\int_{(k/2+1+\epsilon)} \Phi(s)\,ds \\ &= \frac{1}{2\pi i}\sum_{n\geqslant 1} a(n)\int_{(k/2+1+\epsilon)} \Gamma(s)\left(\frac{\sqrt{N}}{2n\pi}\right)^s ds \\ &= \frac{1}{2\pi i}\sum_{n\geqslant 1} a(n)\, e^{-2\pi n/\sqrt{N}} \\ &= \frac{1}{2\pi i} f(i/\sqrt{N}). \end{aligned}$$

This proves part (1) of the theorem. To prove (2), we proceed similarly. As before, let

$$f(z) = \sum_{n\geqslant 1} a(n) q^n$$

be the Fourier expansion of f at infinity, and set $\Delta(s)=\Gamma(s)$ and $P(s)\equiv 1$. Since $\Phi(s)=\Phi(k-s)$, we have $\delta = 0$ and $r=k$ in this context. Just as before, the Dirichlet series

$$\phi(s) = \sum_{n\geqslant 1} a(n)\, e^{-\lambda_n s}$$

is absolutely convergent for $\Re(s)=\sigma > k/2+1$. Since $\Phi(s)$ has an analytic continuation to the entire complex plane, we have

$$\sum_{\sigma_1\leqslant\sigma\leqslant\sigma_2} \mathrm{Res}\{s\Phi(s)\} = 0.$$

Thus, by (6) in Section 4, we have $2iF'(i)+kF(i)=0$. But

$$\begin{aligned} F'(i) &= \frac{1}{2\pi}\int_{(k/2+1+\epsilon)} s\Phi(s)\,ds \\ &= \frac{1}{2\pi}\sum_{n\geqslant 1} a(n)\int_{(k/2+1+\epsilon)} \left(\frac{\sqrt{N}}{2n\pi}\right)^s s\Gamma(s)\,ds \\ &= \frac{1}{2\pi}\sum_{n\geqslant 1} a(n)\int_{(k/2+1+\epsilon)} \Gamma(s+1)\left(\frac{\sqrt{N}}{2n\pi}\right)^s ds \\ &= \frac{1}{\sqrt{N}}\sum_{n\geqslant 1} n a(n)\, e^{-2\pi n/\sqrt{N}} \\ &= \frac{1}{2\pi i\sqrt{N}} f'(i/\sqrt{N}). \end{aligned}$$

Also we have shown that $F(i) = (1/2\pi i) f(i/\sqrt{N})$ in the proof of part (1) of the theorem. Substituting these, we have part (2). □

REMARK. Note that $i/\sqrt{N}$ is a CM point (that is, an algebraic number lying in the upper half plane which generates an imaginary quadratic field). If f is a non-zero modular form of weight k and level N with algebraic Fourier coefficients, any zero of f is either CM or transcendental (see [**20**]).

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