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Transcendental values of the incomplete gamma function and related questions

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Abstract. For $s, x > 0$, the lower incomplete gamma function is defined to be the integral $\gamma(s, x) := \int_0^x t^s e^{-t} \frac{dt}{t}$, which can be continued analytically to an open subset of $\mathbb{C}^2$. Here in this article, we study the transcendence of special values of the lower incomplete gamma function, by means of transcendence of certain infinite series. These series are variants of series which are of great interest in number theory. However, these series are also of independent interest and can be studied in the context of the theory of E-functions.

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1. Introduction. The study of transcendence of infinite series owes its origin to classical mathematics and has been an active area of research for quite some time now. Among the various kinds of infinite series that have been considered and studied in the past, a few specific types of series led us to the series that we consider in this article.

A question of S. Chowla, asked in 1969, and a negative answer to that due to Baker et al. [2], raised the question of the arithmetical nature of the sum $\sum_{n\geq 1} \frac{f(n)}{n}$ where f is a non-zero rational valued prime periodic arithmetical function. The study of the arithmetical nature of such sums was then accomplished by Adhikari et al. [1] in 2001. In 2007, Murty and Saradha [11] employing a different method showed that, when the sum converges, it is in fact transcendental.

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Further in [1,11] the authors investigated the sums of the form $\sum_{n\geq 0}\frac{A(n)}{B(n)}$, where $A(X)$ and $B(X)$ are non-zero polynomials over $\mathbb{Q}$ with $B(X)$ having only simple roots in $\mathbb{Q}\backslash\mathbb{N}$. Here and from now onwards $\mathbb{N}$ denotes the set of all non-negative rational integers. Similar series, where $B(X)$ may have multiple roots, were considered by Pilehrood and Pilehrood [9].

In addition, in [1], and later in [21], the series of the form $\sum_{n\geq 0}\frac{A(n)}{B(n)}z^n$ where $A(X)$, $B(X)$ are non-zero polynomials over $\overline{\mathbb{Q}}$ with $B(X)$ having simple rational roots not in $\mathbb{N}$, and z is algebraic, was examined. Assuming the convergence of such series, its arithmetical nature was established. However, in both the articles, the authors had some restriction, either with respect to the roots of $B(X)$ or with respect to z. The paper [21] also contains results about the transcendental nature of sums of the form $\sum_{n\geq 0}\frac{A(n)f(n)}{B(n)}$ for algebraic valued periodic function f, with some restrictions on the roots of $B(X)$.

In this article we consider variants of such sums and derive analogous results. Let $A(X), B(X)$ be non-zero polynomials over $\overline{\mathbb{Q}}$ with $B(X)$ having no roots in $\mathbb{N}$. For a complex variable z, we consider the power series

$$\sum_{n\geq 0}\frac{A(n)}{B(n)}\frac{z^n}{n!}. \tag{1}$$

Note that such a power series converges for all $z\in\mathbb{C}$. Here we encounter possibilities of studying such series and some of its variants in the context of E-functions (see Section 3 for definition). The arithmetic nature of such sums are then established by the aid of Shidlovskii's theorems [17].

The sums that we consider in this article are also variants of several other functions such as Kummer's function, Bessel functions of the first kind, and many other E-functions, which have been studied by the likes of Siegel [19], Shidlovskii [17], Oleinikov [15], Belogrivov [6,7], Mahler [10]. Many of these results use the works of Shidlovskii about E-functions.

There is a rich theory of E-functions, which has been developed classically in the hands of Siegel, Shidlovskii, Nesterenko etc. and more recently by Beukers. We do not discuss this here as we are interested to derive transcendental results about certain infinite series, putting them in to the context of E-functions.

We show that under certain hypotheses the, sum in (1) is transcendental for non-zero algebraic values of z. To be precise, we prove:

Theorem 1. *Let $A(X), B(X)$ be non-zero polynomials over $\overline{\mathbb{Q}}$ such that $B(X)$ does not divide $A(X)$ and $B(X)$ has rational roots, not belonging to $\mathbb{N}$, with the differences of the roots not in $\mathbb{Z}$. Define,*

$$F(z) := \sum_{n\geq 0}\frac{A(n)}{B(n)}\frac{z^n}{n!}.$$

Then $F(\beta)$ is transcendental, for any non-zero algebraic number β.

As a corollary we establish transcendence of special values of the lower incomplete gamma function. To do so, we need to consider the analytic continuation of $\gamma(s,x)$. For the sake of completeness and future reference, we include a section on the incomplete gamma function (see Section 2).

Corollary 1. *For a non-zero algebraic number β, the special value of the lower incomplete gamma function $\gamma(s,\beta)$ is transcendental for all $s \in \mathbb{Q}\backslash\{0,-1,-2,\dots\}$.*

Remark 1. The interest in special values of the incomplete gamma function has also emerged in the recent theory of harmonic weak Maass forms, introduced by Bruinier and Funke [4] (also see [16] for an elaborate exposition). There, these special values appear in the Fourier expansion of the non-holomorphic part of such Maass forms. The nature of the Fourier coefficients of harmonic weak Maass forms is not completely understood. In certain cases, the arithmetic nature of the Fourier coefficients of the holomorphic part was highlighted in the article [5]. Our work may have application in such studies.

For every harmonic weak Maass form f of weight $2-k$, with $1 < k \in \frac{1}{2}\mathbb{Z}$, on $\Gamma_1(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{Z}) | a \equiv d \equiv 1 \bmod N, c \equiv 0 \bmod N\}$, where N is a positive integer, f has a Fourier expansion of the form

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n,$$

where $q = e^{2\pi i z}$ for $z = x + iy$ with $x, y \in \mathbb{R}$ and $y > 0$ (see [16, Lemma 7.2]). See Sect. 2 for the definition of $\Gamma(s,x)$. One refers

$$f^+(z) := \sum_{n \gg -\infty} c_f^+(n) q^n$$

to be the holomorphic part of f and

$$f^-(z) := \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n$$

to be the non-holomorphic part of f.

As a consequence of Corollary 1, when $k > 1$ is an integer, we have that for all $z = x + iy$ such that $\pi y \in \overline{\mathbb{Q}}$, $\Gamma(k-1, 4\pi|n|y)$ is transcendental.

Note that in Theorem 1, all roots of $B(X)$ were taken to be simple. In the next theorem, we address the case of $B(X)$ having multiple roots. But we restrict ourselves to the case that $B(X)$ has no roots in $\mathbb{Z}$.

Theorem 2. *Let $A(X), B(X)$ be non-zero polynomials over $\overline{\mathbb{Q}}$ with $B(X)$ not dividing $A(X)$ and having rational roots, not belonging to $\mathbb{Z}$, such that the differences of the roots are not in $\mathbb{Z}\backslash\{0\}$. Then for*

$$F(z) := \sum_{n \geq 0} \frac{A(n)}{B(n)} \frac{z^n}{n!},$$

$F(\beta)$ is transcendental, whenever β is a non-zero algebraic number.

Our next two theorems deal with the multiple series analogues of the preceding theorems.

Theorem 3. *Let $r \geq 2$ and $A_i(X), B_i(X)$ be non-zero polynomials over $\overline{\mathbb{Q}}$ such that $B_i(X)$ does not divide $A_i(X)$ for all $1 \leq i \leq r$. Let*

$$S := \{\alpha \in \mathbb{Q} : B_i(\alpha) = 0 \text{ for some } 1 \leq i \leq r\}.$$

Suppose that for each $1 \leq i \leq r$, $B_i(X)$ has simple rational roots which are not in $\mathbb{N}$ and the differences of elements of S are not in $\mathbb{Z}$. Define,

$$G(z_1, \dots, z_r) := \sum_{n_1, \dots, n_r \geq 0} \frac{A_1(n_1) \cdots A_r(n_r)}{B_1(n_1) \cdots B_r(n_r)} \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1! \cdots n_r!}.$$

Then for $\beta_1, \dots, \beta_r \in \overline{\mathbb{Q}}$, linearly independent over $\mathbb{Q}$, $G(\beta_1, \dots, \beta_r)$ is transcendental.

We prove the multiple root analogue of Theorem 3 for a different class of functions.

Theorem 4. *Let $r \geq 2$ and $A_i(X), B(X)$ be non-zero polynomials over $\overline{\mathbb{Q}}$ such that $B(X)$ does not divide $A_i(X)$ for all $1 \leq i \leq r$. Suppose that $B(X)$ has rational roots which are not in $\mathbb{Z}$ and their differences are not in $\mathbb{Z}\backslash\{0\}$. Define,*

$$H(z) := \sum_{n_1, \dots, n_r \geq 0} \frac{A_1(n_1) \cdots A_r(n_r)}{B(n_1) \cdots B(n_r)} \frac{z^{n_1 + \cdots + n_r}}{n_1! \cdots n_r!}.$$

Then for $\beta \in \overline{\mathbb{Q}}\backslash\{0\}$, $H(\beta)$ is transcendental.

For an algebraic valued arithmetical periodic function f, we further consider another natural variant of the series in Theorem 1. In our following theorem we prove:

Theorem 5. *Let $f : \mathbb{N} \to \overline{\mathbb{Q}}$ be a non-zero periodic function and $A(X), B(X)$ be non-zero polynomials over $\overline{\mathbb{Q}}$ such that $B(X)$ does not divide $A(X)$. Define,*

$$F_f(z) := \sum_{n \geq 0} \frac{A(n) f(n)}{B(n)} \frac{z^n}{n!}.$$

Further, assume that $B(X)$ has rational roots, not belonging to $\mathbb{N}$ and with the differences of the roots not in $\mathbb{Z}$. Then $F_f(\beta)$ is transcendental, whenever β is a non-zero algebraic number.

Remark 2. Such a theorem a priori does not fit into the framework of Shidlovskii's theorem. But, we will derive a suitable expression for the above sum so that one can apply Shidlovskii's result.

Remark 3. An analogue of Theorem 3, upon introducing algebraic valued periodic arithmetical functions, will also hold true. For brevity we do not include the exact statement.

2. The incomplete gamma function. We recall, for $s, x > 0$, the *lower incomplete gamma function* is defined as

$$\gamma(s, x) := \int_0^x t^s e^{-t} \frac{dt}{t}.$$

Further, the *upper incomplete gamma function* is defined by

$$\Gamma(s,x) := \int_x^\infty t^s e^{-t} \frac{dt}{t}.$$

Note that, for $s, x > 0$, we thus have, $\Gamma(s) = \gamma(s,x) + \Gamma(s,x)$.

Integrating by parts we obtain the recurrence relation, $\gamma(s,x) = \frac{1}{s}(x^s e^{-x} + \gamma(s+1,x))$. For $n \geq 1$, from the recurrence relation, we get,

$$\gamma(s,x) = \sum_{k=0}^{n} \frac{x^{s+k}e^{-x}}{s(s+1)\cdots(s+k)} + \frac{1}{s\cdots(s+n)}\gamma(s+n+1,x).$$

Note that $e^t > \frac{t^{n-1}}{(n-1)!}$. So,

$$\frac{1}{s\cdots(s+n)}\gamma(s+n+1,x) < \frac{(n-1)!}{s\cdots(s+n)} \int_0^x t^{s+1} dt < \frac{x^{s+2}}{s+n}.$$

Thus for fixed x, $\frac{1}{s\cdots(s+n)}\gamma(s+n+1,x) \to 0$ as $n \to \infty$. Hence we get,

$$\gamma(s,x) = \sum_{k \geq 0} \frac{x^{s+k}e^{-x}}{s(s+1)\cdots(s+k)} = x^s e^{-x}\Gamma(s) \sum_{k\geq 0} \frac{x^k}{\Gamma(s+k+1)}. \tag{2}$$

Let us define

$$\gamma^*(s,z) := e^{-z} \sum_{k \geq 0} \frac{z^k}{\Gamma(s+k+1)}.$$

Note that as a power series in z, the above series converges for all $z \in \mathbb{C}$. Hence as a function of z, $\gamma^*(s,z)$ is an entire function. Further for fixed z,

$$\left| \frac{z^k}{\Gamma(s+k+1)} \right| = \frac{|z|^k}{|(s+k)\cdots(s+1)||\Gamma(s+1)|} \leq \frac{|z|^k}{k!|\Gamma(s+1)|}.$$

Now $\frac{1}{\Gamma(s)}$ is an entire function, hence on a compact subset of $\mathbb{C}$, it is bounded. Thus on every compact subset of $\mathbb{C}$, the series is uniformly convergent, for a fixed z. Since $\frac{1}{\Gamma(s)}$ is an entire function and the series converges uniformly on compact subsets of $\mathbb{C}$, we get that for a fixed z, $\gamma^*(s,z)$ is an entire function as a function of s. In this context we recall a theorem of Hartogs by which we get $\gamma^*(s,z)$ is holomorphic on $\mathbb{C}^2$.

Theorem 6. (Hartogs) (See [13, p. 43]) *Let f be a complex valued function defined on an open set $\Omega \subset \mathbb{C}^n$. For $\mathbf{a} = (a_1,\ldots,a_n) \in \mathbb{C}^n$ and $1 \leq i \leq n$, let us define $\Omega_{i,\mathbf{a}} := \{z \in \mathbb{C} : (a_1,\ldots,a_{i-1},z,a_{i+1},\ldots,a_n) \in \Omega\}$ and $f_{i,\mathbf{a}} : \Omega_{i,\mathbf{a}} \to \mathbb{C}$ by $f_{i,\mathbf{a}}(z) = f(a_1,\ldots,a_{i-1},z,a_{i+1},\ldots,a_n)$. Now suppose, for any $\mathbf{a} \in \mathbb{C}^n$ and $1 \leq i \leq n$, $f_{i,\mathbf{a}}$ is holomorphic on $\Omega_{i,\mathbf{a}}$. Then f is holomorphic on Ω.*

Now using (2), we can define,

$$\gamma(s,z) := z^s\Gamma(s)\gamma^*(s,z),$$

for $z \in \mathbb{C}$ such that $-\pi < \arg(z) < \pi$ and $s \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$. Hence on the open set $U := \{(z_1,z_2) \in \mathbb{C}^2 : z_1 \neq 0,-1,-2,\ldots \text{ and } -\pi < \arg(z_2) < \pi\}$,

$\gamma(s,z)$ is well-defined and holomorphic. Further, $\Gamma(s,z)$ can also be defined on U as a holomorphic function, by setting, $\Gamma(s,z)=\Gamma(s)-\gamma(s,z)$.

We now derive another series expansion of $\gamma(s,z)$ which will be used to prove Corollary 1. We again assume that $s,x>0$. Now,

$$\gamma(s,x)=\int_0^x t^{s-1}e^{-t}dt=\int_0^x \sum_{k\geq 0}(-1)^k\frac{t^{s+k-1}}{k!}\,dt=x^s\sum_{k\geq 0}(-1)^k\frac{x^k}{k!(s+k)}.$$

Equivalently, using (2) we get,

$$\gamma^*(s,x)=\frac{1}{\Gamma(s)}\sum_{k\geq 0}(-1)^k\frac{x^k}{k!(s+k)}.$$

This series again converges for all values of $x\in\mathbb{C}$. For s in a compact subset of $\mathbb{C}$, the series is uniformly convergent as a function of s also. Hence for $(s,z)\in U$ we can write,

$$\gamma(s,z)=z^s\sum_{k\geq 0}(-1)^k\frac{z^k}{k!(s+k)}. \tag{3}$$

3. Preliminaries. In 1929, Siegel [19] introduced a generalization of the exponential function and called it an *E-function*. More precisely, we have:

Definition 1. A function $f(z)=\sum_{n\geq 0}c_n\frac{z^n}{n!}$ is said to be an E-function if

(1) There exists a number field K such that $c_n\in K$ for all $n\in\mathbb{N}$,
(2) For any $\epsilon>0$, $\overline{|c_n|}=O(n^{\epsilon n})$ as $n\to\infty$, where for an algebraic number $\alpha\in K$, $\overline{|\alpha|}$ is called the absolute height of α and is defined to be the maximum of the modulus of conjugates of α in K,
(3) For any $\epsilon>0$, there exists a sequence $\{q_n\}_{n\geq 1}$ of positive rational integers such that q_nc_k are algebraic integers for $k=0,1,\ldots,n$; $n\geq 1$ and that $q_n=O(n^{\epsilon n})$ as $n\to\infty$.

In the above context, it is also important to fix an embedding of K into $\mathbb{C}$. Note that, from condition (2) in the above definition, we get that E-functions are entire functions. Let us denote the space of all E-functions by $\mathbf{E}$. Then it can be seen that $\mathbf{E}$ is a $\overline{\mathbb{Q}}$-algebra which is closed under differentiation, anti-differentiation, and the change of variable z to αz for any $\alpha\in\overline{\mathbb{Q}}$. For an elaborate account on E-functions we refer the reader to [14, 18, 20].

As far as the recent literature is concerned, it has now become customary to use the following definition to define an E-function. We refer the reader to [3].

Definition 2. An entire function $f(z)$ given by a power series $f(z)=\sum_{n\geq 0}c_n\frac{z^n}{n!}$ is called an E-function if

(1) $c_n\in\overline{\mathbb{Q}}$ for all $n\in\mathbb{N}$,
(2) f satisfies a linear differential equation $Ly=0$ with coefficients in $\overline{\mathbb{Q}}[z]$,
(3) $h(c_0,c_1,\ldots,c_k)=O(k)$ for all k, where $h(c_0,c_1,\ldots,c_k)$ denotes the maximum of the logarithms of the absolute heights of $c_0,c_1,\ldots,c_k$.

Note that the condition (2) in the definition due to Siegel translates to the condition that $h(c_0, c_1, \ldots, c_k) = o(k \log k)$. Here in this article, we will consider three types of E-functions. For $\lambda \neq -1, -2, \ldots$, let us define,

$$\varphi_\lambda(z) := \sum_{n \geq 0} \frac{z^n}{(\lambda+1)\cdots(\lambda+n)}.$$

Note that, $\varphi_0(z) = e^z$. For $\lambda \in \mathbb{Q}$ it can be shown that $\varphi_\lambda(z)$ is an E-function.

For a power series $f(z) = \sum_{n\geq 0} a_n z^n$, we denote its anti-derivative $\sum_{n \geq 0} a_n \frac{z^{n+1}}{n+1}$ by $\int^z f(t)dt$. It can be verified that,

$$\varphi_\lambda(z) = \lambda z^{-\lambda} e^z \int^z t^{\lambda-1} e^{-t} dt. \tag{4}$$

Our next set of E-functions are defined as follows:

$$\omega_\lambda(z) := \sum_{n \geq 0} \frac{z^n}{n!(\lambda+n)}; \qquad \text{for } \lambda \neq 0, -1, -2, \ldots \tag{5}$$

It is then easy to note that, $\omega_\lambda(z) = z^{-\lambda} \int^z t^{\lambda-1} e^t dt$. Hence we obtain,

$$\omega_\lambda(z) = \frac{1}{\lambda} e^z \varphi_\lambda(-z). \tag{6}$$

For these E-functions Shidlovskii proved:

Theorem 7. (See [18, p. 193] or [17]) *Suppose that* $\lambda_0 \in \mathbb{N}; \lambda_1, \ldots, \lambda_n (n \geq 0) \in \mathbb{Q}$ *such that* $\lambda_i - \lambda_j \notin \mathbb{Z}, 1 \leq i < j \leq n; \alpha_1, \ldots, \alpha_m (m \geq 1) \in \overline{\mathbb{Q}}$ *which are linearly independent over* $\mathbb{Q}$*; and* $\xi_1, \ldots, \xi_m \in \overline{\mathbb{Q}}$ *are distinct and non-zero. Then the* $(n+1)m$ *numbers* $\varphi_{\lambda_0}(\alpha_i)$ *and* $\varphi_{\lambda_j}(\xi_i), 1 \leq i \leq m, 1 \leq j \leq n$, *are algebraically independent.*

Now we consider another set of E-functions for $\lambda_k \in \mathbb{Q}$, $\lambda_k \neq 0, -1, -2, \ldots$. For $k = 0$, set $\psi_0(z) = e^z$ and for $k \geq 1$ we define,

$$\psi_{\lambda_k}(z) := \sum_{n \geq 0} \frac{z^n}{n!(\lambda_1+n)\cdots(\lambda_k+n)}. \tag{7}$$

The following theorem due to Shidlovskii about these E-functions is the key ingredient of the proof of Theorem 2.

Theorem 8. (See [18, p. 239] or [17]) *Suppose that* $\lambda_k \in \mathbb{Q}\backslash\mathbb{Z}$ *for* $k = 1, \ldots, m$*; the differences* $\lambda_i - \lambda_k \notin \mathbb{Z}\backslash\{0\}$*; and* $\xi \in \overline{\mathbb{Q}}\backslash\{0\}$*. Then the* $m+1$ *numbers*

$$e^\xi, \psi_{\lambda_1}(\xi), \ldots, \psi_{\lambda_m}(\xi)$$

are algebraically independent.

4. Proofs of the Theorems. *Proof of Theorem 1.* Let $-\alpha_1, \ldots, -\alpha_d$ be the roots of $B(X)$. If $\deg A(X) < \deg B(X)$, by partial fractions we can write,

$$\frac{A(X)}{B(X)} = \sum_{i=1}^{d} \frac{c_i}{X+\alpha_i},$$

where $c_i \in \overline{\mathbb{Q}}$ for all $1 \le i \le d$ and not all of them are zero. Using (5) and (6) we get,

$$F(z) = \sum_{i=1}^{d} c_i\, \omega_{\alpha_i}(z) = e^z \sum_{i=1}^{d} \frac{c_i}{\alpha_i}\varphi_{\alpha_i}(-z) = \varphi_0(z) \sum_{i=1}^{d} \frac{c_i}{\alpha_i}\varphi_{\alpha_i}(-z).$$

Now by Theorem 7, the following $d+1$ numbers, $\varphi_0(\beta), \varphi_{\alpha_1}(-\beta), \ldots, \varphi_{\alpha_d}(-\beta)$ are algebraically independent. Since $c_i \neq 0$ for some i, we get that $F(\beta)$ is transcendental.

Now suppose $\deg A(X) \ge \deg B(X)$. Since $B(X)$ does not divide $A(X)$, by the division algorithm in $\overline{\mathbb{Q}}[X]$, we write $A(X) = P(X)B(X) + Q(X)$, where $\deg Q(X) < \deg B(X)$. So in this case we have,

$$F(z) = \sum_{n\ge 0} P(n)\frac{z^n}{n!} + \sum_{n\ge 0} \frac{Q(n)}{B(n)}\frac{z^n}{n!}.$$

By induction on r, it can be easily verified that,

$$\sum_{n\ge 0} n^r \frac{z^n}{n!} = P_r(z)e^z$$

for some polynomial $P_r(X) \in \mathbb{Z}[X]$. Hence

$$\sum_{n\ge 0} P(n)\frac{z^n}{n!} = \tilde{P}(z)e^z$$

for some polynomial $\tilde{P}(X) \in \overline{\mathbb{Q}}[X]$. Therefore we get,

$$F(z) = \tilde{P}(z)\varphi_0(z) + \varphi_0(z) \sum_{i=1}^{d} \frac{d_i}{\alpha_i}\varphi_{\alpha_i}(-z),$$

for some $d_i \in \overline{\mathbb{Q}}$, not all zero. So, by Theorem 7, we get $F(\beta)$ is transcendental. $\square$

Remark 4. The proof also suggests that, if we start with $\beta_1, \ldots, \beta_m \in \overline{\mathbb{Q}}$ such that they are linearly independent over $\mathbb{Q}$, then we can get that $F(\beta_1), \ldots, F(\beta_m)$ are algebraically independent.

Remark 5. We cannot drop the condition that $B(X)$ does not divide $A(X)$, otherwise we would have $F(z) = \tilde{P}(z)\varphi_0(z)$, which is equal to zero for any zero of $\tilde{P}$.

Proof of Corollary 1. The lower incomplete gamma function has the expansion

$$\gamma(s, z) = \sum_{n\ge 0} \frac{(-1)^n}{n!}\frac{z^{s+n}}{s+n}.$$

So we can write $\gamma(s, z) = z^s F(-z)$, where $F(z) := \sum_{n\ge 0} \frac{1}{s+n}\frac{z^n}{n!}$. It follows from Theorem 1 that for $s \in \mathbb{Q}\backslash\{0, -1, -2, \ldots\}$ and $\beta \in \overline{\mathbb{Q}}\backslash\{0\}$, $\gamma(s, \beta)$ is transcendental. $\square$

Remark 6. The referee rightly pointed out that the Theorem 3 in page 192 of [18] also gives result about transcendental values of the lower incomplete gamma function. But, since we obtain our result as a corollary of Theorem 1 we have less restriction on the hypothesis.

Remark 7. To avoid the problem of $\gamma(s, z)$ being multi-valued, we chose a specific branch of z^s. We defined $\gamma(s, z)$ as a well-defined holomorphic function on the open set $U = \{(z_1, z_2) \in \mathbb{C}^2 : -\pi < \arg(z_1) < \pi \text{ and } z_2 \neq 0, -1, -2, \ldots\}$. However, for $\beta \in \overline{\mathbb{Q}} \backslash \{0\}$ such that $\arg(\beta) \notin (-\pi, \pi)$, we get the same result because β^s is algebraic when $s \in \mathbb{Q}$, independent of the choice of branch of z^s.

Remark 8. At present we have very little knowledge about values of the gamma function at rational numbers. For example, we know that $\Gamma(1/2) = \sqrt{\pi}$; $\Gamma(1/3), \Gamma(1/4)$ are transcendental; but the nature of $\Gamma(1/5)$ is unknown. However some partial result is known due to Grinspan. Grinspan [8] showed that at least two of the three numbers $\pi, \Gamma(1/5), \Gamma(2/5)$ are algebraically independent. Hence we can say at least one among $\Gamma(1/5), \Gamma(2/5)$ is transcendental. In a recent article [12], Murty and Weatherby obtained a result concerning transcendental values of the Gamma function at various CM points, for example: the Gamma function takes transcendental values at rational points on the imaginary axis. Hence as of now we can only say that, $\Gamma(s, z)$ is transcendental for $s \in \mathbb{N} \backslash \{0\}$ and $z \in \overline{\mathbb{Q}} \backslash \{0\}$.

Proof of Theorem 3. Because of uniform convergence on compact subsets of $\mathbb{C}^r$, we have,

$$G(z_1, \ldots, z_r) = \prod_{i=1}^{r} \sum_{n_i \geq 0} \frac{A_i(n_i)}{B_i(n_i)} \frac{z_i^{n_i}}{n_i!}.$$

Let $S_i := \{\alpha_{ij} : 1 \leq j \leq d_i\}$ be the set the roots of $B_i(X)$ for $1 \leq i \leq r$. Then as in Theorem 1 we can deduce that,

$$\sum_{n_i \geq 0} \frac{A_i(n_i)}{B_i(n_i)} \frac{z_i^{n_i}}{n_i!} = \tilde{P}_i(z_i)\varphi_0(z_i) + \varphi_0(z_i) \sum_{j=1}^{d_i} \nu_{ij} \varphi_{\alpha_{ij}}(-z_i),$$

for some $\nu_{ij} \in \overline{\mathbb{Q}}$. Therefore,

$$G(\beta_1, \ldots, \beta_r) = \prod_{i=1}^{r} \left(\tilde{P}_i(\beta_i)\varphi_0(\beta_i) + \varphi_0(\beta_i) \sum_{j=1}^{d_i} \nu_{ij} \varphi_{\alpha_{ij}}(-\beta_i) \right),$$

which is non-zero by Theorem 1 and hence transcendental by Theorem 7. □

Proof of Theorem 5. The proof of Theorem 1 was completed by using the simple idea of partial fraction decomposition to connect the corresponding infinite series to the E-functions that we mentioned in Sect. 3 and by appealing to Shidlovskii's theorem. Here we would further need the idea of using Fourier inversion to put the corresponding series into the framework of Shidlovskii's theorem.

Let q be the period of f. Now by Fourier inversion we can write

$$f(n) = \sum_{b=0}^{q-1} \hat{f}(b)\zeta_q^{nb}.$$

Hence, we get

$$F_f(z) = \sum_{n\geq 0}\sum_{b=0}^{q-1} \hat{f}(b)\zeta_q^{nb}\frac{A(n)}{B(n)}\frac{z^n}{n!} = \sum_{b=0}^{q-1}\hat{f}(b)\sum_{n\geq 0}\frac{A(n)}{B(n)}\frac{(z\zeta_q^b)^n}{n!} = \sum_{b=0}^{q-1}\hat{f}(b)F(z\zeta_q^b),$$

where F is as in Theorem 1. So for β algebraic,

$$F_f(\beta) = \sum_{b=0}^{q-1}\hat{f}(b)\left(\tilde{P}(\beta\zeta_q^b)\varphi_0(\beta\zeta_q^b) + \varphi_0(\beta\zeta_q^b)\ \sum_{i=1}^{d}\frac{d_i}{\alpha_i}\varphi_{\alpha_i}(-\beta\zeta_q^b)\right).$$

Note that the set $\{\beta\zeta_q^b : 0 \leq b \leq q-1\}$ is not $\mathbb{Q}$-linearly independent. Hence one cannot readily appeal to Theorem 7 to conclude the theorem. We get around this difficulty by the following observation: one can work with a maximal linearly independent subset, and which is possible due to the fact that $\exp(x+y) = \exp(x)\exp(y)$. We elaborate below.

Let us start with a maximal $\mathbb{Q}$-linearly independent subset, say $\{\beta_1, \ldots, \beta_m\}$. For $0 \leq b \leq q-1$, we write, $\beta\zeta_q^b = \sum_{j=1}^m r_{bj}\beta_j$ with $r_{bj} \in \mathbb{Q}$. Now for a fixed j, one can write, $r_{bj} = r_j \cdot n_{bj}$ for all $0 \leq b \leq q-1$, where r_j is a rational number and the n_{bj}'s are integers. Thus $\varphi_0(\beta\zeta_q^b) = \prod_{j=1}^m \varphi_0(r_{bj}\beta_j) = \prod_{j=1}^m \varphi_0(r_j\beta_j)^{n_{bj}}$. Hence we get,

$$F_f(\beta) = \sum_{b=0}^{q-1}\hat{f}(b)\left(\tilde{P}(\beta\zeta_q^b)\prod_{j=1}^{m}\varphi_0(r_j\beta_j)^{n_{bj}} + \prod_{j=1}^{m}\varphi_0(r_j\beta_j)^{n_{bj}}\ \sum_{i=1}^{d}\frac{d_i}{\alpha_i}\varphi_{\alpha_i}(-\beta\zeta_q^b)\right).$$

Now we can apply Theorem 7 and conclude the theorem, provided at least one of the algebraic coefficients involved in the expression is non-zero. Since $B(X)$ does not divide $A(X)$, we get $d_{i_0} \neq 0$ for some $1 \leq i_0 \leq d$. Further, f is non-zero, thus $\hat{f}(b_0)$ is non-zero for some $0 \leq b_0 \leq q-1$. Thus we get that the coefficient of $\varphi_{\alpha_{i_0}}(-\beta\zeta_q^{b_0})\prod_{j=1}^m \varphi_0(r_j\beta_j)^{n_{b_0 j}}$ is non-zero. □

Proof of Theorem 2. In this case, unlike Theorem 1, a partial fraction decomposition does not help us to connect our series to the relevant E-functions. Rather, the following lemma enables us to view our series in terms of the E-functions that we encounter in Sect. 3. We view this lemma as a variant of the partial fraction decomposition.

Lemma 1. *Let K be an algebraically closed field and $A(X), B(X) \in K[X]$ with the condition* $\deg A < \deg B$. *Let $\alpha_1, \ldots, \alpha_d$ be the roots of $B(X)$. Then, we can write,*

$$\frac{A(X)}{B(X)} = \sum_{k=1}^{d}\frac{c_k}{\prod_{i=1}^{k}(X-\alpha_i)},$$

for some $c_k \in K$.

Proof of the Lemma. Without loss of generality let $A(X), B(X)$ be monic. We induct on the degree of A.

If $\deg A = 0$, the result is immediate.

Suppose $A = \prod_{i=1}^{r}(X-\beta_i)$, then

$$\begin{aligned}\frac{A(X)}{B(X)} &= \frac{(X-\alpha_d+\alpha_d-\beta_r)\prod_{i=1}^{r-1}(X-\beta_i)}{\prod_{i=1}^{d}(X-\alpha_i)} \\ &= \frac{\prod_{i=1}^{r-1}(X-\beta_i)}{\prod_{i=1}^{d-1}(X-\alpha_i)} + \frac{(\alpha_d-\beta_r)\prod_{i=1}^{r-1}(X-\beta_i)}{\prod_{i=1}^{d}(X-\alpha_i)}.\end{aligned}$$

So we are done by induction since the degrees of the polynomials in the numerators are less than $\deg A$. □

Now we start the proof of Theorem 2. Since $B(X)$ does not divide $A(X)$, by the division algorithm in $\overline{\mathbb{Q}}[X]$ we write, $A(X) = P(X)B(X) + Q(X)$, where $\deg Q(X) < \deg B(X)$. As in Theorem 1, we get,

$$F(z) = \tilde{P}(z)e^z + \sum_{n\geq 0}\frac{Q(n)}{B(n)}\frac{z^n}{n!}.$$

Now using (7) and Lemma 1, $F(z)$ can be re-written as

$$F(z) = \tilde{P}(z)e^z + \sum_{k=1}^{d} c_k\psi_{\alpha_k}(z),$$

where, $-\alpha_1, \ldots, -\alpha_d$ are the roots of $B(X)$ and c_k's are in $\overline{\mathbb{Q}}$, not all are zero. Hence by Theorem 8, $F(\beta)$ is transcendental. □

Proof of Theorem 4. Again by uniform convergence on compact subsets of $\mathbb{C}$, we have,

$$H(z) = \prod_{i=1}^{r}\sum_{n_i\geq 0}\frac{A_i(n_i)}{B(n_i)}\frac{z^{n_i}}{n_i!}.$$

Now, following the steps of Theorem 2 we can write,

$$H(z) = \prod_{i=1}^{r}\left(\tilde{P}_i(z)e^z + \sum_{k=1}^{d} c_{ik}\psi_{\alpha_k}(z)\right).$$

Hence we have that $H(\beta)$ is non-zero by Theorem 2 and therefore transcendental by Theorem 8. □

5. Concluding remarks. Clearly, this work opens up several avenues of future research, for example,

(1) Can φ_λ be an E-function, for some $\lambda \in \overline{\mathbb{Q}}\backslash\mathbb{Q}$? Most likely not. What about other functions that we considered here?
(2) In our theorems, can one have roots which are not necessarily rational?
(3) What about a multiple root analogue of Theorem 5 like other theorems? Clearly, Theorem 8 is not good enough to obtain such a result.

(4) What if we replace a periodic function with a multiplicative or an additive function or more generally with a nice arithmetical function?

These questions we relegate to future work.

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