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ALGEBRAIC INDEPENDENCE OF VALUES OF MODULAR FORMS

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We investigate values of modular forms with algebraic Fourier coefficients at algebraic arguments. As a consequence, we conclude about the nature of zeros of such modular forms. In particular, the singular values of modular forms (that is, values at CM points) are related to the recent work of Nesterenko. As an application, we deduce the transcendence of critical values of certain Hecke L-series. We also discuss how these investigations generalize to the case of quasi-modular forms with algebraic Fourier coefficients.

Keywords: Transcendental values of modular forms; Schneider's theorem; Nesterenko's theorem.

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1. Introduction

The naturally occurring transcendental functions like the exponential function and the logarithm function take transcendental values when evaluated at algebraic points, except for some obvious exceptions. This is also exhibited by the Weierstrass-$\wp$ function associated to an elliptic curve defined over algebraic number fields. We also expect other transcendental functions like the gamma function and Riemann zeta function to exhibit similar properties.

In this paper, we investigate this phenomena for modular forms and quasi-modular forms which are a rich source of transcendental functions. This work is an attempt to investigate along the above suggested supposition, relying on the works of Schneider, Bertrand, Chudnovsky and Nesterenko.

We first introduce some preliminaries relevant to our results in the next section. In Sec. 2, we take up the theme of the nature of zeros of modular forms. Investigations of such zeros have been carried out by Rankin and Swinnerton-Dyer [15], Kanou [10], Kohnen [11] and Gun [8]. Throughout the paper, $\mathfrak{H}$ denotes the upper half plane and a CM point is an element of $\mathfrak{H}$ lying in an imaginary quadratic field. Every modular form considered is assumed to be non-zero.

To study the algebraic nature of values taken by modular forms, we need to define an equivalence relation on the set of all modular forms with algebraic Fourier coefficients. We define two such modular forms f and g to be equivalent, denoted by $f \sim g$, if there exists natural numbers k_1, k_2 such that $f^{k_2} = \lambda g^{k_1}$ with $\lambda \in \overline{\mathbb{Q}}^*$.

We have the following theorem.

Theorem 1. *Let f be a non-zero modular form of weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Suppose that the Fourier coefficients of f are algebraic. Then any zero of f is either* CM *or transcendental.*

Let Δ be the unique normalized cusp form of weight 12 for the full modular group. Then the above theorem extends to the following:

Theorem 2. *Let f be as in Theorem 1 not equivalent to Δ and $\alpha \in \mathfrak{H}$ be an algebraic number such that $f^{12}(\alpha)/\Delta^k(\alpha)$ is algebraic. Then α is necessarily a* CM *point.*

Remark 1.1. If f is equivalent to Δ and α is CM, then $f(\alpha)$ is transcendental by a theorem of Schneider (see [14, p. 141], for instance). On the other hand, if $\alpha \in \mathfrak{H}$ is non-CM algebraic, a conjecture of Nesterenko (see [13, p. 31]) will imply the transcendence of $f(\alpha)$.

While investigating the nature of values of modular forms at algebraic numbers in $\mathfrak{H}$, it is natural to divide them into CM points and non-CM points. In Sec. 3, we investigate the values taken by modular forms at CM points. Here, we prove the following theorem.

Theorem 3. *Let $\alpha \in \mathfrak{H}$ be such that $j(\alpha) \in \overline{\mathbb{Q}}$. Then $e^{2\pi i\alpha}$ and $\Delta(\alpha)$ are algebraically independent.*

An algebraic α for which $j(\alpha)$ is algebraic is a CM point. In this case, $\Delta(\alpha)$ can be explicitly expressed as a power of period of an elliptic curve defined over $\overline{\mathbb{Q}}$. This can be inferred from the papers of Ramachandra, Siegel and Schneider which we briefly describe later.

As a consequence of the above theorem, we now have the following:

Theorem 4. *Let $\alpha \in \mathfrak{H}$ be such that $j(\alpha) \in \overline{\mathbb{Q}}$. Then for a non-zero modular form f for $\mathrm{SL}_2(\mathbb{Z})$ with algebraic Fourier coefficients, $f(\alpha)$ is algebraically independent with $e^{2\pi i\alpha}$ except when $f(\alpha) = 0$.*

We note that there exist transcendental numbers α for which $j(\alpha)$ is algebraic. This is a consequence of CM theory and surjectivity of the j function.

For a non-CM algebraic number, we have the following theorem:

Theorem 5. *Let $\alpha \in \mathfrak{H}$ be a non-*CM *algebraic number. Let S_α be the set of all non-zero modular forms of arbitrary weight for $\mathrm{SL}_2(\mathbb{Z})$ with algebraic Fourier coefficients for which $f(\alpha)$ is algebraic. Then, up to equivalence, S_α has at most one element.*

The existence of the fugitive exceptional class in the above theorem can be ruled out if we assume the conjecture of Nesterenko alluded to in the remark above.

All of these theorems extend to higher levels. For the sake of clarity of exposition, and to explicate the new ingredients that are necessary to treat the higher level case, we have decided to treat this case separately in the last section. We prove:

Theorem 6. *If f is a non-zero modular form (of weight k and level N) with algebraic Fourier coefficients. Then the following statements are true:*

(1) *Any zero of f is either* CM *or transcendental.*
(2) *If $\alpha \in \mathfrak{H}$ is algebraic and $f \not\sim \Delta$ such that $f^{12}(\alpha)/\Delta^k(\alpha)$ is algebraic, then α is necessarily a* CM *point.*
(3) *If $j(\alpha)$ is algebraic, then $f(\alpha)$ and $e^{2\pi i\alpha}$ are algebraically independent unless $f(\alpha) = 0$.*
(4) *If for a fixed $\alpha \in \mathfrak{H}$ algebraic and not* CM, *we let S_α be the set of all non-zero modular forms of arbitrary weight and level N, with algebraic Fourier coefficients such that $f(\alpha)$ is algebraic, then up to equivalence, S_α has at most one element.*

In the penultimate section, we consider the values taken by quasi-modular forms. Here we prove the following theorem.

Theorem 7. *Let $\alpha \in \mathfrak{H}$ be such that the number $j(\alpha)$ is algebraic. Then for any quasi-modular form f for $\Gamma_0(\mathrm{N})$ with algebraic Fourier coefficients, the numbers $f(\alpha)$ and $e^{2\pi i\alpha}$ are algebraically independent unless $f(\alpha) = 0$.*

We also observe that nearly all of these results have obvious generalization to half-integral weight modular forms with algebraic Fourier coefficients. Indeed, the square of any such form is then of integral weight to which nearly all our theorems apply.

Finally, in the last section we deduce the transcendence of critical values of certain Hecke L-series as an application.

2. Preliminaries

We begin by fixing some notations. For $z \in \mathfrak{H}$, we have the following functions

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)e^{2\pi inz},$$

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)e^{2\pi inz},$$

$$E_6(z) = 1 - 504\sum_{n=1}^{\infty} \sigma_5(n)e^{2\pi inz},$$

where $\sigma_k(n) = \sum_{d|n} d^k$. We also have the j-function given by

$$j(z) = 1728\,\frac{E_4(z)^3}{E_4(z)^3 - E_6(z)^2}\,.$$

It is known, from classical theory of complex multiplication, that if $z \in \mathfrak{H}$ is a CM point, then $j(z)$ is an algebraic number, lying in the Hilbert class field of $\mathbb{Q}(z)$. For instance, we have $j(i) = 1728$ while $j(\rho) = 0$ where $\rho = e^{2\pi i/3}$.

For algebraic points in the upper half plane, Schneider ([17]) in 1937 proved the following result:

Theorem 8 ([17]). *If $z \in \mathfrak{H}$ is algebraic, then $j(z)$ is algebraic if and only if z is* CM*.*

Much later, Chudnovsky ([3], see also [4]) in 1976 showed that if $z \in \mathfrak{H}$, then at least two of the numbers $E_2(z)$, $E_4(z)$, $E_6(z)$ are algebraically independent. Chudnovsky's theorem proves that $\Gamma(1/3)$ and $\Gamma(1/4)$ are transcendental. In 1995, Barré-Sirieix *et al.* ([1]) made a breakthrough in transcendence theory by proving the long-standing conjecture of Mahler and Manin according to which the modular invariant $J(e^{2\pi iz}) := j(z)$ assumes transcendental values at any non-zero complex (or p-adic) algebraic number $e^{2\pi iz}$ in the unit disc. Note that such a z is necessarily transcendental. Finally, Nesterenko ([12]) provided a fundamental advance by generalizing both the results of Chudnovsky and Barré–Sirieix–Diaz–Gramain–Philibert.

Theorem 9 ([12]). *Let z be a point in the upper half plane. Then at least three of the four numbers*

$$e^{2\pi iz}, \quad E_2(z), \quad E_4(z), \quad E_6(z)$$

are algebraically independent.

We note that the result of Schneider does not follow from the theorem of Nesterenko. As pointed out by Nesterenko ([13, p. 31]), both his as well as

Schneider's theorem will follow from the following conjecture:

Conjecture 10. *Let z be a point in the upper half plane and assume that at most three of the following five numbers*

$$z, \quad e^{2\pi iz}, \quad E_2(z), \quad E_4(z), \quad E_6(z)$$

are algebraically independent. Then z is necessarily a CM *point and the field*

$$\mathbb{Q}(e^{2\pi iz}, E_2(z), E_4(z), E_6(z))$$

has transcendence degree 3.

3. Zeros of Modular Forms and Proofs of Theorems 1 and 2

Kanou ([10]) showed that for even $k \geq 16$, the normalized Eisenstein series E_k has at least one transcendental zero in $\mathfrak{H}$. Soon after, Kohnen ([11]) proved that any zero of E_k in $\mathfrak{H}$ different from ρ or i is necessarily transcendental. We now proceed to prove Theorem 1 which considers zeros of arbitrary modular forms.

Proof of Theorem 1. Let f be a non-zero modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with algebraic Fourier coefficients. Let $g(z)$ be the function defined as

$$g(z) = \frac{f^{12}(z)}{\Delta^k(z)},$$

where $\Delta(z)$ is the Ramanujan cusp form of weight 12. Thus $g(z)$ is a modular function of weight 0 and hence is a rational function in j. Since Δ does not vanish on $\mathfrak{H}$, g is a polynomial in j. Further, since f has algebraic Fourier coefficients, $g(z) = P(j(z))$, where $P(x)$ is a polynomial with algebraic coefficients. If α is a zero of f, then $P(j(\alpha)) = 0$ and hence $j(\alpha)$ is algebraic. Thus by Schneider's theorem, α is either CM or transcendental. This completes the proof. $\square$

We note that the above theorem does not say anything about the transcendental zeros of f. However, when f is the Eisenstein series E_k, we have some more information about the location of their zeros. For instance, all the zeros of E_k in the fundamental domain were shown to lie in the arc

$$\{e^{i\theta} \mid \pi/2 \leq \theta \leq 2\pi/3\}$$

by Rankin and Swinnerton-Dyer ([15]).

It is worthwhile to point out that for cusp forms, the situation is rather different. Here we have a result due to Rudnick [16] which is as follows: let $\{f_k\}$ be a sequence of L^2-normalized holomorphic cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ such that f_k is of weight k, the order of vanishing of f_k at the cusp is $o(k)$, and the masses $y^k|f_k(z)|^2 dV(z)$ (where $dV(z)$ stands for the normalized hyperbolic measure on the fundamental domain) tend in the weak star topology to $c\,dV(z)$ for some constant $c > 0$. Then the zeros of f_k (in the fundamental domain) are equidistributed with respect to $dV(z)$.

The proof of Theorem 2 proceeds exactly along the same lines as the proof of Theorem 1. Indeed, as before we deduce that $j(\alpha)$ is algebraic and by Schneider's theorem, this means that α is a CM point. This completes the proof of Theorem 2.

Let us observe that this theorem does not tell us anything about the possible transcendence of $f(\alpha)$ when α is an arbitrary algebraic number. However, when α is a CM point and more generally when $j(\alpha)$ is algebraic, we can conclude algebraic independence of $f(\alpha)$ and $e^{2\pi i\alpha}$. We do this in the next section.

4. Proofs of Theorems 3 and 4

We begin with the proof of Theorem 3. Since $j(\alpha)$ is algebraic, $\Delta(\alpha)$ is transcendental. For, algebraicity of $\Delta(\alpha)$ will imply that $j(\alpha)\Delta(\alpha) = E_4(\alpha)^3$ is algebraic and hence both $E_4(\alpha)$ and $E_6(\alpha)$ are algebraic. This will contradict Chudnovsky's theorem. Now suppose that $e^{2\pi i\alpha} = q$ and $\Delta(\alpha)$ are algebraically dependent. Since $\Delta(\alpha)$ is transcendental, there exists a non constant polynomial $P(X) = \sum_i p_i X^i$ where p_i's are polynomials in $\Delta(\alpha)$ with algebraic coefficients such that $P(q) = 0$. Thus q is algebraic over the field $\overline{\mathbb{Q}}(E_4(\alpha), E_6(\alpha))$. Since $j(\alpha)$ is algebraic, transcendence degree of $\overline{\mathbb{Q}}(E_4(\alpha), E_6(\alpha))$ is one which is also the transcendence degree of $\overline{\mathbb{Q}}(E_4(\alpha), E_6(\alpha), q)$. This will contradict Nesterenko's theorem.

We now give the proof of Theorem 4. Suppose that $f(\alpha)$ is not equal to zero. Since the non-zero number $f^k(\alpha)/\Delta^{12}(\alpha)$ is a polynomial in $j(\alpha)$ with algebraic coefficients, it is algebraic. Thus the fields $\overline{\mathbb{Q}}(q, f(\alpha))$ and $\overline{\mathbb{Q}}(q, \Delta(\alpha))$ have the same transcendence degree and hence the theorem follows from Theorem 3.

The study of $\Delta(z)$ when z is a CM point is a chapter in the theory of complex multiplication and is treated in several places, for example in the famous paper of Chowla and Selberg ([2]). In this paper (see Lemma 3, p. 109), they prove that if $z_1, z_2 \in \mathfrak{H}$ belong to the same imaginary quadratic field, then $\Delta(z_1)/\Delta(z_2)$ is an algebraic number. Thus if $\alpha \in \mathbb{Q}(\sqrt{-D})$ with $D > 0$, the transcendence of $\Delta(\alpha)$ is equivalent to the transcendence of $\Delta\big(\frac{D+\sqrt{-D}}{2}\big)$. Computation of this number is carried out in number of places, for instance [14, p. 142], we find that

$$\Delta\left(\frac{D+\sqrt{-D}}{2}\right) = (-1)^D \left(\frac{2\pi}{d}\right)^6 \prod_{a=1}^{d} \Gamma\left(\frac{a}{d}\right)^{3\chi_d(a)\omega/h}.$$

Here, d is the absolute discriminant of $\mathbb{Q}(\sqrt{-D})$ and $\chi_d(a)$ is the Kronecker–Jacobi symbol $(\frac{d}{a})$. We shall call the factor

$$\Omega := \prod_{a=1}^{d} \Gamma\left(\frac{a}{d}\right)^{\chi_d(a)},$$

the Chowla–Selberg period of level D. Thus Δ at a CM point is, up to an algebraic factor, a product of π^6 and a power of Ω. As mentioned by Ramachandra (see [14, p. 141]), the transcendence of $\Delta(\sqrt{-D})$ follows from a theorem of Schneider ([20, p. 89]).

5. Modular Forms at Non-CM Algebraic Points and Proof of Theorem 5

Let f and g be modular forms in S_α of weight k_1 and k_2, respectively. Let α be a non-CM algebraic number in $\mathfrak{H}$ and suppose that both $f(\alpha)$ and $g(\alpha)$ be algebraic. Note that by Theorem 1, neither is equal to zero. We consider the modular form $F = f^{k_2}(\alpha)g^{k_1} - g^{k_1}(\alpha)f^{k_2}$ of weight k_1k_2. If $F \not\equiv 0$, then by Theorem 1, any zero of this modular form is either CM or transcendental. Since α is non-CM and algebraic, we get a contradiction unless F is identically zero. This means that f and g are equivalent (in the sense defined in Sec. 1).

6. The Higher Level Case and the Proof of Theorem 6

To treat the higher level case, the essential idea is to use the fact that the field of modular functions of level N with algebraic Fourier coefficients (with respect to $e^{2\pi iz/N}$) is a finite Galois extension of $\overline{\mathbb{Q}}(j)$. This is essentially a consequence of [18, Proposition 6.9]. We indicate a proof of this assertion. If our function f has algebraic Fourier coefficients, then as is well-known f^σ is again a modular form with algebraic Fourier coefficients for every automorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (here f^σ is the modular form obtained by applying σ to every Fourier coefficient in the q expansion of f). Since the Fourier coefficients of f lie in a fixed algebraic extension of $\mathbb{Q}$, f has only finitely many such distinct conjugate forms. We consider $g^\sigma = (f^\sigma)^{12}/\Delta^k$ and define the polynomial

$$\prod_{\gamma\in\Gamma(1)\backslash\Gamma(N)} \prod_\sigma (X - g^\sigma \mid \gamma)$$

which is a polynomial whose coefficients are weight zero modular functions of level 1 with algebraic Fourier coefficients. Here, the outer product is over coset representatives of $\Gamma(1)\backslash\Gamma(N)$. Consequently, we can write the polynomial as

$$\sum_{m=0}^{d} a_m(j)X^m, \quad a_d(j) = 1,$$

where $a_m(j)$ are polynomials in j. In particular, the constant term is a polynomial in j. If α is algebraic such that $f(\alpha) = 0$, then $a_0(j(\alpha)) = 0$ and again by Schneider's theorem, α is a CM point. This establishes part (1) of Theorem 6. Proofs of rest of the parts follow analogously, suitably modifying the proofs for level one.

7. Values of Quasi-Modular Forms and Proof of Theorem 7

We begin by recalling the notion of quasi-modular forms, following Kaneko and Zagier ([9]). As defined by them, an *almost-holomorphic* modular form of weight k

is a function $F(z)$ with the same transformation properties and growth conditions as modular forms, but which belongs to the ring $\mathcal{C}[[q]][Y^{-1}]$ instead of $\mathcal{C}[[q]]$ and is of the form

$$F(z) = \sum_{m=0}^{M} f_m(z) Y^{-m}, \quad f_0, \dots, f_M \text{ holomorphic}$$

where $M \geq 0$ is an integer not exceeding $k/2$. The holomorphic function $f_0(z)$ obtained formally as the constant term with respect to Y^{-1} of F is called a *quasi-modular form* of weight k. The simplest example of a quasi-modular form is E_2. The ring of quasi-modular forms are closed under differentiation and hence all derivatives of modular forms are quasi-modular. Further, in the above mentioned paper, they proved the following:

Theorem 11. *Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$. Any quasi-modular form on Γ can be written uniquely as a polynomial in E_2 with coefficients which are modular forms for Γ.*

We now proceed to prove Theorem 7. We are given that $\alpha \in \mathfrak{H}$ is such that $j(\alpha)$ is algebraic. Then $E_4(\alpha)$ and $E_6(\alpha)$ are algebraically dependent. Let f be a non-zero quasi-modular form for $\mathrm{SL}_2(\mathbb{Z})$ with algebraic Fourier coefficients and $f(\alpha) \neq 0$. By the result of [9], f is a polynomial in E_2, E_4 and E_6, that is,

$$f(z) = P(E_2, E_4, E_6),$$

where $P(X, Y, Z)$ is a polynomial with algebraic coefficients since f has algebraic Fourier coefficients. More precisely,

$$f(z) = f_0(z) + f_1(z)E_2(z) + \cdots + f_r(z)E_2(z)^r$$

where $f_i(z)$ is a modular form of weight $k - 2i$. Note that $f(\alpha)$ is transcendental. For, either $f(\alpha) = f_0(\alpha) \notin \overline{Q}$ or otherwise the transcendence degree of the field $\overline{Q}(E_2(\alpha), E_4(\alpha), E_6(\alpha))$ is one. This will contradict Chudnovsky's result. Now if $f(\alpha)$ and $e^{2\pi i\alpha}$ are algebraically dependent, since $f(\alpha)$ is transcendental, there exists a non constant polynomial $P(X) = \sum_i p_i X^i$ where p_i's are polynomials in $f(\alpha)$ with algebraic coefficients such that $P(q) = 0$. Thus q is algebraic over the field $\overline{\mathbb{Q}}(E_2(\alpha), E_4(\alpha), E_6(\alpha))$. Since $j(\alpha)$ is algebraic, transcendence degree of $\overline{\mathbb{Q}}(E_2(\alpha), E_4(\alpha), E_6(\alpha), q)$ is at most two. This will contradict Nesterenko's theorem.

Now for a quasi-modular form f for $\Gamma_0(\mathrm{N})$ with algebraic Fourier coefficients, the proof follows by noting that:

(1) A modular function of weight zero for $\Gamma_0(\mathrm{N})$ is a rational function in $j(z)$ and $j(Nz)$.
(2) If $j(\alpha)$ is algebraic, then any rational function in $j(\alpha)$ and $j(N\alpha)$ with algebraic coefficients is also algebraic. Thus for any modular form g for $\Gamma_0(\mathrm{N})$ with algebraic Fourier coefficients, $g(\alpha)$ is algebraic over $Q(\Delta(\alpha))$.

8. Transcendence of Critical Values of Certain Hecke *L*-Series

For an imaginary quadratic field K, Shimura ([19]) introduced the Dirichlet series

$$L^r(s;\beta,\mathfrak{b}) = \sum_{\substack{\xi\in\beta+\mathfrak{b}\\ \xi\neq 0}} \xi^{-r}|\xi|^{r-2s},$$

where $\mathfrak{b}$ is a lattice in $K, \beta\in K$ and r is a natural number. He shows that

$$\pi^{-s}\Gamma\left(s+\frac{r}{2}\right)L^r(s;\beta,\mathfrak{b})$$

extends to an entire function (see [19, Theorem 7.3]). In [19, Theorem 13.2], he proves:

Theorem 12. *Let j be an integer such that $2-r\le j\le r$ and $r-j\in 2\mathbb{Z}$. Let $\sigma\in K\cap\mathfrak{H}$ and g a modular form of weight r with Fourier coefficients in the maximal abelian extension $\mathbb{Q}_{\mathrm{ab}}$ of $\mathbb{Q}$ such that $g(\sigma)\neq 0$. Then*

$$L^r(j/2;\beta,\mathfrak{b})\in\pi^{-(r+j)/2}g(\sigma)K_{\mathrm{ab}}$$

where K_{ab} is the maximal abelian extension of K.

As a consequence of our work discussed in earlier sections, we deduce that $L^r(j/2;\beta,\mathfrak{b})$ is transcendental and in fact of the form $\pi^a\Omega^b$ up to an algebraic factor, where Ω is the Chowla–Selberg period attached to K and a, b are rational numbers.

If we now fix an integral ideal $\mathcal{C}$ and a non-zero integer r, let λ be a Hecke character such that

$$\lambda((\beta)) = |\beta|^r/\beta^r,\quad \forall\,\beta\in\mathcal{O}_K,\quad \beta \text{ coprime to } \mathcal{C}.$$

Then, for any integer j with $2-r\le j\le r$, $r-j\in 2\mathbb{Z}$, $\tau\in K\cap\mathfrak{H}$, we have $L(j/2,\lambda)$ is an algebraic number times $\pi^{(r+j)/2}h(\tau)$ where h is a modular form of weight r with algebraic Fourier coefficients and $h(\tau)\neq 0$ (see [19, Theorem 13.6]). As a consequence of our work, we deduce

Theorem 13. *Let K, $\mathcal{C}$ and λ be as above, and D be the discriminant of K. For any integer j with $2-r\le j\le r$, $r-j\in 2\mathbb{Z}$, the special value $L(j/2,\lambda)$ is of the form $\pi^a\Omega^b$ up to an algebraic factor, where Ω is the Chowla–Selberg period of K. The values $e^{\pi\sqrt{D}}$, $L(j/2,\lambda), \pi$ are algebraically independent. In particular, $L(j/2,\lambda)$ is transcendental for every value of j in the stated interval.*

We close with some final remarks. The special values $L(j/2,\lambda)$ fit into a larger philosophy of Deligne ([7]) regarding "critical values" of L-series arising in the "motivic" context. The explicit evaluation of $L(j/2,\lambda)$ was carried out by Damerell [5, 6] as part of his doctoral thesis. One expects these critical values to be transcendental and the results of this paper are one step more towards the realization of such an expectation.

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