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TRANSCENDENTAL SUMS RELATED TO THE ZEROS OF ZETA FUNCTIONS

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Abstract. While the distribution of the non-trivial zeros of the Riemann zeta function constitutes a central theme in Mathematics, nothing is known about the algebraic nature of these non-trivial zeros. In this article, we study the transcendental nature of sums of the form

$$\sum_{\rho} R(\rho) x^{\rho},$$

where the sum is over the non-trivial zeros ρ of $\zeta(s)$, $R(x) \in \overline{\mathbb{Q}}(x)$ is a rational function over algebraic numbers and $x > 0$ is a real algebraic number. In particular, we show that the function

$$f(x) = \sum_{\rho} \frac{x^{\rho}}{\rho}$$

has infinitely many zeros in $(1, \infty)$, at most one of which is algebraic. The transcendence tools required for studying $f(x)$ in the range $x < 1$ seem to be different from those in the range $x > 1$. For $x < 1$, we have the following non-vanishing theorem: If for an integer $d \geqslant 1$, $f(\pi\sqrt{d}x)$ has a rational zero in $(0, 1/\pi\sqrt{d})$, then

$$L'(1, \chi_{-d}) \neq 0,$$

where χ_{-d} is the quadratic character associated with the imaginary quadratic field $K := \mathbb{Q}(\sqrt{-d})$. Finally, we consider analogous questions for elements in the Selberg class. Our proofs rest on results from analytic as well as transcendental number theory.

§1. *Introduction.* For $s \in \mathbb{C}$ with $\Re(s) > 1$, the Riemann zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is well known that $\zeta(s)$ has an analytic continuation to the entire complex plane except at $s = 1$, where it has a simple pole with residue 1. The functional equation for $\zeta(s)$ is determined by the equation

$$\xi(s) = \xi(1 - s),$$

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where $\xi(s)$ is an entire function defined as

$$\xi(s) := \frac{s(s-1)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The values taken by the Riemann zeta function at positive integers and as well as its location of zeros have been studied extensively since the time of Euler and Riemann. However the nature of these special values continues to elude us though there has been some success following the works of Apery, Beukers, Rivoal and Zudilin among others.

On the other hand, the nature of the non-trivial zeros of the Riemann zeta function is as mysterious as the special values. Let $\mathbb{X}$ denote the set of its non-trivial zeros. Nothing is known about this set vis-a-vis transcendence. For instance, consider the field $F = \mathbb{Q}(\mathbb{X})$. Then one can ask the following question:

Is the transcendence degree of F over $\mathbb{Q}$ at least one?

Also, consider the $\mathbb{Q}$-vector space V (inside $\mathbb{C}$) generated by the imaginary parts of elements in $\mathbb{X}$. Again, we can ask the seemingly easier question:

Is the dimension of V over $\mathbb{Q}$ at least two?

In fact, there is a folklore conjecture that the imaginary parts of the non-trivial zeros in the upper half plane are linearly independent over $\mathbb{Q}$ (see, for example, Theorem A of [**9**], see also [**16**]).

One believes that the answers to both these questions should be affirmative, but it is not clear if the answers lie within the reach of the existing transcendence tools or we need to discover new tools. One of the obstacles to answer such basic questions is that the Riemann zeta function does not satisfy any differential equation with algebraic parameters. More precisely, a classical result of Voronin [**21**] asserts that $\zeta(s)$ does not satisfy any equation of the form

$$\sum_{j=0}^{n} s^j F_j(\zeta(s), \ldots, \zeta^{(n-1)}(s)) = 0$$

for all s lying on a line $\Re(s) = \sigma$ with $\sigma \in (1/2, 1)$. Here the F_j for $j = 0, \ldots, n$ are continuous functions on $\mathbb{C}^n$, not all identically zero. This functional independence of Riemann zeta function renders effete the applicability of the known general transcendental tools to the question of the nature of non-trivial zeta zeros.

The goal of this note is to study the nature of certain general sums related to the zeros of the ζ-function. More generally, we also consider sums related to the zeros of functions in the Selberg class.

Let us now introduce the type of sums we are interested in. Throughout the paper $\overline{\mathbb{Q}}$ will denote the field of algebraic numbers in $\mathbb{C}$. Let $A, B \in \overline{\mathbb{Q}}[t]$ be polynomials. We study the transcendental nature of sums of the form

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)} x^{\rho}, \tag{1}$$

where the sum is over the non-trivial zeros ρ of $\zeta(s)$. Here $x > 0$ is a real number. In this paper, we study the situation where $B(t)$ has only simple zeros.

As shall be evident, we need to consider the cases $x > 1$, $x = 1$ and $0 < x < 1$ separately. It appears that the transcendence input in the study of the case $x > 1$ is different from that of $x < 1$. The case $x = 1$ is perhaps more mysterious. For instance, if the rational function $R(x) = A(x)/B(x)$ satisfies the functional equation $R(x) = -R(1-x)$, then

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)} = \frac{1}{2} \sum_{\rho} \left(\frac{A(\rho)}{B(\rho)} + \frac{A(1-\rho)}{B(1-\rho)} \right) = 0,$$

since the functional equation for $\zeta(s)$ implies that ρ is a zero if and only if $1-\rho$ is a zero.

The study of sums involving zeros of the Riemann zeta function can have deep arithmetic significance. For instance, in 1997, Li [12] obtained a simple criterion (now known as Li's criterion) linking positivity of certain sums to the Riemann hypothesis. More precisely, let

$$\lambda_n := \sum_{\rho} \left(1 - \left(1 - \frac{1}{\rho} \right)^n \right).$$

Then the Riemann hypothesis is true if and only if $\lambda_n \geqslant 0$ for every natural number n (see also work of Brown [2]). This result has led to a flurry of activity and a plethora of interesting results have emerged from this. For example, Bombieri and Lagarias [1] derived the following elegant arithmetic identity. Define the *Stieltjes constants* γ_n by

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n.$$

It is not difficult to show that the γ_n are given by the limits

$$\gamma_n = \lim_{m \to \infty} \left(\sum_{k=1}^{m} \frac{\log^n k}{k} - \frac{\log^{n+1} m}{n+1} \right),$$

and these can be viewed as generalizations of the more familiar Euler constant $\gamma_0 = \gamma$. This allows us to define the related constants η_n via

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \eta_n (s-1)^n.$$

By long division, it is now clear that the η_n can be expressed as polynomials in the γ_n with rational coefficients. For instance,

$$\eta_0 = -\gamma_0, \qquad \eta_1 = -\gamma_1 + \tfrac{1}{2}\gamma_0^2, \qquad \eta_3 = -\gamma_2 + \gamma_0\gamma_1 - \tfrac{1}{3}\gamma_0^3.$$

Then, it is shown in [1] that

$$\lambda_n = -\sum_{j=1}^{n} \binom{n}{j} \eta_j + 1 - (\log 4\pi + \gamma)\frac{n}{2} - \sum_{j=2}^{n} (-1)^{j-1} \binom{n}{j} (1 - 2^{-j})\zeta(j).$$

For $n = 1$, this reduces to

$$\sum_{\rho} \frac{1}{\rho} = 1 + \frac{\gamma}{2} - \frac{\log 4\pi}{2} = 0.0230957\ldots \tag{2}$$

In this context, one has the following curious equivalence:

$$\text{Riemann hypothesis} \iff \sum_{\rho} \frac{1}{|\rho|^2} = 2 + \gamma - \log 4\pi.$$

This is easy to show. Indeed,

$$2\sum_{\rho} \frac{1}{\rho} = \sum_{\rho} \left(\frac{1}{\rho} + \frac{1}{\bar{\rho}} \right) = \sum_{\rho} \frac{2\Re(\rho)}{|\rho|^2} \tag{3}$$

from which we immediately see the result if the Riemann hypothesis is true. For the converse, suppose $\rho = x + iy$, with $x, y \in \mathbb{R}$ is a zero such that $x > 1/2$. Then, $(1 - x)^2 + y^2 < x^2 + y^2$ so that

$$(x - 1/2)\left(\frac{1}{(1 - x)^2 + y^2} - \frac{1}{x^2 + y^2} \right) > 0.$$

In other words, for $x > 1/2$,

$$\frac{1}{2}\left(\frac{1}{(1 - x)^2 + y^2} + \frac{1}{x^2 + y^2} \right) > \frac{x}{x^2 + y^2} + \frac{1 - x}{(1 - x)^2 + y^2}. \tag{4}$$

Writing our sum as

$$\sum_{\rho} \frac{1}{|\rho|^2} = \sum_{\rho, \Re(\rho) = 1/2} \frac{1}{|\rho|^2} + \sum_{\rho, \Re(\rho) \neq 1/2} \frac{1}{|\rho|^2}$$

and pairing the zero ρ with $1 - \rho$ in the second sum, we deduce from (4) that

$$\sum_{\Re(\rho) \neq 1/2} \frac{1}{|\rho|^2} > 2 \sum_{\Re(\rho) \neq 1/2} \frac{\Re(\rho)}{|\rho|^2}.$$

Since

$$\sum_{\rho, \Re(\rho) = 1/2} \frac{1}{|\rho|^2} = 2 \sum_{\Re(\rho) \neq 1/2} \frac{\Re(\rho)}{|\rho|^2}$$

we see that if the Riemann hypothesis is false, then using (3) and (2),

$$\sum_{\rho} \frac{1}{|\rho|^2} > \sum_{\rho} \frac{2\Re(\rho)}{|\rho|^2} = 2 + \gamma - \log 4\pi$$

contrary to our hypothesis. This idea can be easily generalized to the Selberg class (see [5] for details).

Related to this, we study the possible transcendental nature of sums of the form

$$\sum_{\nu>0} \frac{\cos(\nu \log x)}{\frac{1}{4} + \nu^2}$$

for any algebraic $x > 1$, subject to Riemann hypothesis. Here, the sum is over the positive imaginary parts of the non-trivial zeros of the zeta function. Similar investigations are also carried out without the assumption of the Riemann hypothesis.

The expression for λ_n has been studied by several authors from various angles. Coffey [3] writes

$$\lambda_n = 1 - \frac{n}{2}(\gamma + \log 4\pi) + S_1(n) + S_2(n),$$

where

$$S_1(n) = \sum_{j=2}^{n} \binom{n}{j} (-1)^j \left(1 - \frac{1}{2^j}\right)\zeta(j)$$

and

$$S_2(n) = \sum_{j=1}^{n} \binom{n}{j} \eta_{j-1}.$$

Coffey showed that for $n \geqslant 2$,

$$\tfrac{1}{2}(n(\log n + \gamma - 1) + 1) \leqslant S_1(n) \leqslant \tfrac{1}{2}(n(\log n + \gamma + 1) - 1).$$

In particular, $S_1(n)$ is non-negative for every $n \geqslant 2$. This theorem reduces the study of λ_n to the study of $S_2(n)$ and sums involving the Stieltjes constants.

Bombieri and Lagarias [1] show that the condition of positivity can be considerably weakened to deduce the Riemann hypothesis. In fact, they show that if for any $\epsilon > 0$, there is a constant $c(\epsilon) > 0$ such that

$$\lambda_n \geqslant -c(\epsilon)\mathrm{e}^{\epsilon n}$$

for every $n \geqslant 1$, then the Riemann hypothesis follows. Estimates for the Stieltjes constants have been studied by several authors (see, for example, [4]), but these estimates give super-exponential estimates for the sums in question.

Though the prototypical zeta function is the Riemann zeta function, it is useful and interesting to consider the more general setting of the Selberg class $\mathcal{S}$ which we carry out in this paper.

Before we proceed further, let us fix some notations. Throughout the paper, we denote by ρ_F or sometimes by ρ (if the context is clear) the non-trivial zeros of an element $F(s)$ in the Selberg class. In this context, we examine sums of the form (1) when ρ runs through zeros of a fixed element of the Selberg class. More details of this theory can be found in §6 below.

§2. *Some transcendental prerequisites.* First we recall the following theorem due to Alan Baker which will play a key role in our investigation.

THEOREM 2.1 (Baker). *If $\alpha_1, \ldots, \alpha_m$ are non-zero algebraic numbers such that* $\log\alpha_1, \ldots, \log\alpha_m$ *are linearly independent over* $\mathbb{Q}$*, then*

$$1, \log\alpha_1, \ldots, \log\alpha_m$$

are linearly independent over $\overline{\mathbb{Q}}$.

Let $\mathcal{L}$ denote the $\overline{\mathbb{Q}}$-vector space generated by the logarithms of non-zero algebraic numbers. We refer to this as the space of Baker periods. Baker's theorem asserts that every non-zero Baker period is transcendental.

We shall call the elements in the $\overline{\mathbb{Q}}$-vector space generated by the logarithms of non-zero algebraic numbers and 1 as *extended Baker periods.*

We now recall the following far reaching conjecture in transcendence theory due to Schanuel.

SCHANUEL'S CONJECTURE. *Suppose that $\alpha_1, \ldots, \alpha_n$ are complex numbers which are linearly independent over $\mathbb{Q}$. Then the transcendence degree of the field*

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n})$$

over $\mathbb{Q}$ is at least n.

We shall need the following consequence of the above conjecture which is not difficult to deduce. This was done in an earlier work of ours (see [**8**] for details).

PROPOSITION 2.2. *Assume that Schanuel's conjecture is true. If $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers such that* $\log\alpha_1, \ldots, \log\alpha_n$ *are linearly independent over* $\mathbb{Q}$*, then*

$$\log\alpha_1, \ldots, \log\alpha_n, \log\pi$$

are algebraically independent. In particular, $\log\pi$ *is a transcendental number which is not an extended Baker period.*

Finally, we shall need the following theorem of Nesterenko (see [**17**] and [**18**]).

THEOREM 2.3. *Let $\wp(z)$ be a Weierstrass $\wp$-function with algebraic invariants g_2, g_3 and with complex multiplication by an order of an imaginary quadratic field K. Let ω be a non-zero period and η the corresponding quasi-period. Then for any $\tau \in K$ with $\Im(\tau) \neq 0$, each of these sets*

$$\{\pi, \omega, \mathrm{e}^{2\pi\mathrm{i}\tau}\} \quad \textit{and} \quad \{\omega, \eta, \mathrm{e}^{2\pi\mathrm{i}\tau}\}$$

is algebraically independent over $\mathbb{Q}$.

§3. *The case of the Riemann zeta function.* We begin by considering the following function

$$f : (0, \infty) \to \mathbb{C}$$

given by

$$f(x) = \sum_{\rho} \frac{x^{\rho}}{\rho} := \lim_{T\to\infty} \sum_{|t|<T} \frac{x^{\rho}}{\rho},$$

where $\rho = \sigma + it$ runs over the non-trivial zeros of the Riemann zeta function in the critical strip $0 < \Re(s) < 1$. Recall that

$$f(1) = \tfrac{1}{2}\gamma + 1 - \tfrac{1}{2}\log 4\pi,$$

where γ is the Euler's constant. It is not known whether $f(1)$ is an irrational number.

We are interested in studying the values taken by the function f at algebraic points. We first have the following theorem.

THEOREM 3.1. *The set X given by*

$$X := \{f(x) : x \in (1, \infty) \cap \overline{\mathbb{Q}}\}$$

has at most one algebraic element.

Proof. For $x > 1$, consider the function $\psi_0(x)$ given by

$$\psi_0(x) = \begin{cases} \sum_{n\leqslant x} \Lambda(n) & \text{if } x \text{ is not a prime power,} \\ \sum_{n\leqslant x} \Lambda(n) - \dfrac{1}{2}\Lambda(x) & \text{otherwise.} \end{cases}$$

Observe that $\psi_0(x)$ is a Baker period. Here

$$\Lambda(n) = \begin{cases} \log p & \text{if } n \text{ is a power of a prime number } p, \\ 0 & \text{otherwise.} \end{cases}$$

is the classical von Mangoldt function.

When $x > 1$, one has the following explicit formula of von Mangoldt (see [10, p. 77], for instance)

$$f(x) = x - \psi_0(x) - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

For $x > 1$, consider the function

$$g(x) = f(x) - x + \log 2\pi = -\psi_0(x) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

Note that $g(x)$ is a strictly decreasing function in $(1, \infty)$. Now suppose that $f(x)$ is algebraic at two distinct algebraic points, say α and β. Then

$$g(\alpha) - g(\beta) = f(\alpha) - \alpha - f(\beta) + \beta$$

is a non-zero algebraic number. But this is also a Baker period, a contradiction. □

THEOREM 3.2. *The function f has infinitely many zeros in $(1, \infty)$ of which at most one is algebraic.*

Proof. Note that for any algebraic $x > 1$, $f(x) - x + \log 2\pi$ is a Baker period. Thus by Baker's theorem, f is injective on the set of algebraic numbers greater than 1. Hence f can have at most one algebraic zero.

We now show that f has infinitely many zeros in $(1, \infty)$. Let us write f as

$$f(x) = h(x) - \psi_0(x),$$

where $h(x) = x - \frac{1}{2}\log(1 - 1/x^2) - \log 2\pi$.

Since $\psi_0(n) - n = \Omega_{\pm}(n^{1/2})$ (see [10, p. 91]), the sequence $\{f(n)\}_{n \in \mathbb{N}}$ changes sign infinitely often. In particular, there exists infinitely many $N \in \mathbb{N}$ for which $f(N) < 0$ while $f(N+1) > 0$. Note that f is continuous in $(1, \infty)$ except at prime powers where it is right continuous. Also $h(x)$ is a strictly increasing continuous function in $(1, \infty)$. Let $N_0 > 1$ be a natural number such that $f(N_0) < 0$ while $f(N_0+1) > 0$. If $f(x) \geqslant 0$ for some $x \in (N_0, N_0+1)$, we have a zero of f in (N_0, N_0+1). Assume otherwise. Since $\psi_0(x)$ is non-negative and constant in $[N_0, N_0+1)$, N_0+1 cannot be a prime power. Thus the function f is continuous in $[N_0, N_0+1]$ and hence must have a zero in this interval. □

We also have the following conditional result.

THEOREM 3.3. *Assume Schanuel's conjecture. Then X has no algebraic element.*

Proof. Suppose that both $f(x)$ and x are algebraic. Then $\log \pi$ lies in the $\overline{\mathbb{Q}}$-vector space generated by logarithms of non-zero algebraic numbers and 1. But by Proposition 2.2, this is not possible if we assume Schanuel's conjecture. □

We now consider the case for $0 < x < 1$. When $0 < x < 1$, one has the following expression as indicated by (Ingham [10, p. 81]):

$$\sum_{n \leqslant 1/x}{}' \frac{\Lambda(n)}{n} = -\log x - \gamma + \sum_{\rho} \frac{x^{\rho}}{\rho} + \frac{1}{2}\log\frac{1+x}{1-x} - x,$$

where γ denotes Euler's constant. The dash in the sum means that there is a correction factor of $1/2$ in the last term of the sum involving the von Mangoldt function when x is the reciprocal of some prime power.

The above expression can be deduced by considering the following integral

$$\frac{1}{2\pi \mathrm{i}} \int_{3-\mathrm{i}\infty}^{3+\mathrm{i}\infty} \frac{x^{1-s}}{1-s} \frac{\zeta'}{\zeta}(s)\, ds.$$

By Perron's formula, this integral is equal to the left-hand side of the above expression. As with the explicit formula for $x > 1$, completing this integral into a rectangular contour, we will have contributions exactly from the residues of the poles of $(x^{1-s}/(1-s))(\zeta'/\zeta)(s)$ in the complex plane. The double pole at $s = 1$ contributes the factor $-\log x - \gamma$. The poles from non-trivial zeros contribute the factor

$$\sum_{\rho} \frac{x^{1-\rho}}{1-\rho} = \sum_{\rho} \frac{x^{\rho}}{\rho}.$$

Finally, the trivial zeros of the zeta function contribute

$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} = \frac{1}{2} \log \frac{1+x}{1-x} - x.$$

Arguing as earlier, we can now deduce the following result.

THEOREM 3.4. *The set Y given by*

$$Y := \{f(x) - x : x \in (0, 1) \cap \overline{\mathbb{Q}}\}$$

has at most one algebraic element. In particular, f has at most one algebraic zero in $(0, 1)$.

Proof. For $0 < x < 1$,

$$f(x) - x - \gamma = \sum_{n \leqslant 1/x}{}' \frac{\Lambda(n)}{n} + \log x - \frac{1}{2} \log \frac{1+x}{1-x}$$

and hence a Baker period if x is algebraic. If there are two algebraic values in the set, we argue as we did in our earlier theorem. □

Also, we immediately observe the following.

COROLLARY 3.5. *If $f(x)$ is algebraic for some algebraic x in* $(0, 1)$*, then γ is transcendental.*

It seems that the existence of the (presumably) fictitious algebraic element in the above theorem cannot be ruled out under Schanuel's conjecture. Thus the transcendence tools required for studying $f(x)$ in the range $x < 1$ seem to be different from those in the range $x > 1$. We however have the following curious theorem.

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THEOREM 3.6. *For an integer $d \geqslant 1$, suppose that $f(\pi\sqrt{d}x)$ has a rational zero in $(0, 1/\pi\sqrt{d})$. Then for the quadratic character χ_{-d} associated with the imaginary quadratic field $K := \mathbb{Q}(\sqrt{-d})$, one has*

$$L'(1, \chi_{-d}) \neq 0.$$

Proof. As discussed above for $0 < x < 1$,

$$f(x) = \sideset{}{'}\sum_{n \leqslant 1/x} \frac{\Lambda(n)}{n} + \log x + \gamma - \frac{1}{2}\log\frac{1+x}{1-x} + x.$$

Suppose $f(\pi\sqrt{d}r) = 0$ for some rational $r \in (0, 1/\pi\sqrt{d})$. Write $x = \pi r\sqrt{d}$. Then we have

$$\gamma = -\sideset{}{'}\sum_{n \leqslant 1/x} \frac{\Lambda(n)}{n} - \log x + \frac{1}{2}\log\frac{1+x}{1-x} - x.$$

Therefore,

$$e^{2\gamma} = \alpha e^{-2\pi r\sqrt{d}}, \tag{5}$$

where $\alpha \in \overline{\mathbb{Q}}(\pi)$.

Now let χ_{-d} be the quadratic character associated to the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ such that $L'(1, \chi_{-d}) = 0$.

It is known (see [**14**, p. 848]) that (essentially by the Chowla–Selberg formula)

$$\exp\left(\frac{L'(1, \chi_{-d})}{L(1, \chi_{-d})} - \gamma\right) = (2D/A^2)\prod_{a=1}^{D}\Gamma(a/D)^{-\chi_{-d}(a)w/2h}$$

where $A = \sqrt{D/\pi}$, D is the absolute discriminant of K, h and w are the class number and order of unit group of K respectively.

As observed by Gross [**7**], the number

$$\prod_{a=1}^{D}\Gamma(a/D)^{\chi_{-d}(a)}$$

is, up to an algebraic factor, equal to a product of a power of π and a power of a non-zero period ω of the CM elliptic curve attached to the full ring of integers of K.

Since $L'(1, \chi_{-d}) = 0$, we see from above that $e^{\gamma} \in \overline{\mathbb{Q}}(\pi, \omega)$. On the other hand, from (5), we have $e^{2\gamma} = \alpha e^{-2\pi r\sqrt{d}}$ with $\alpha \in \overline{\mathbb{Q}}(\pi)$. This contradicts Nesterenko's result (Theorem 2.3). □

As evident, while $\log\pi$ is the mysterious number that shows up in the evaluation of $f(x)$ for $x > 1$, it is γ that enters the picture for $x < 1$. We would like to obtain transcendence results involving both $\log \pi$ and γ. For $x > 1$, replacing x by $1/x$ in Ingham's formula, we obtain

$$L(x) := \sum_{n \leqslant x}' \frac{\Lambda(n)}{n} = \log x - \gamma + \sum_{\rho} \frac{x^{-\rho}}{\rho} - \frac{1}{2} \log \frac{x+1}{x-1} - \frac{1}{x}.$$

Recall that for such an x, we have

$$\psi_0(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2} \log\left(1 - \frac{1}{x^2}\right).$$

Assuming the Riemann hypothesis so that a typical zero ρ is of the form $1/2+\mathrm{i}\nu$, and pairing the zeros $1/2+\mathrm{i}\nu$ with $1/2-\mathrm{i}\nu$, we obtain the following expression for $x > 1$:

$$\begin{aligned}\sum_{\nu>0} \frac{2\cos(\nu \log x)}{1/4+\nu^2} &= \sum_{\nu} \frac{x^{\mathrm{i}\nu} + x^{-\mathrm{i}\nu}}{1/2+\mathrm{i}\nu} \\ &= \frac{x - \psi_0(x)}{\sqrt{x}} - \frac{\log 2\pi}{\sqrt{x}} - \frac{1}{2\sqrt{x}} \log\left(1 - \frac{1}{x^2}\right) \\ &\quad + \sqrt{x}(L(x) - \log x) + \gamma\sqrt{x} - \frac{\sqrt{x}}{2} \log \frac{x+1}{x-1} + \frac{1}{\sqrt{x}}\end{aligned}$$

and hence

$$\begin{aligned}\sum_{\nu>0} \frac{2\cos(\nu \log x)}{1/4+\nu^2} + \frac{\log 2\pi}{\sqrt{x}} - \gamma\sqrt{x} &= \frac{x - \psi_0(x)}{\sqrt{x}} - \frac{1}{2\sqrt{x}} \log\left(1 - \frac{1}{x^2}\right) \\ &\quad + \sqrt{x}(L(x) - \log x) - \frac{\sqrt{x}}{2} \log \frac{x+1}{x-1} + \frac{1}{\sqrt{x}}.\end{aligned}$$

We now have the following.

THEOREM 3.7. *Assume the Riemann hypothesis. For any algebraic $x > 1$,*

$$\sum_{\nu>0} \frac{2\cos(\nu \log x)}{1/4+\nu^2} + \frac{\log 2\pi}{\sqrt{x}} - \gamma\sqrt{x}$$

is an extended Baker period. If Schanuel's conjecture is true, then the following set

$$\left\{\sum_{\nu>0} \frac{\cos(\nu \log x)}{1/4+\nu^2} : x > 1, x \in \overline{\mathbb{Q}}\right\}$$

has at most one algebraic number.

We now derive a related result without assuming the Riemann hypothesis. To this end, we observe that we can write

$$\sum_{\rho} \frac{x^{-\rho}}{\rho} = \sum_{\rho} \frac{x^{-(1-\rho)}}{1-\rho} = \sum_{\rho} \frac{x^{\rho-1}}{1-\rho},$$

by virtue of the functional equation. Thus

$$L(x) = \log x - \gamma - \sum_{\rho} \frac{x^{\rho-1}}{\rho} + \left(\sum_{\rho} \frac{x^{\rho-1}}{\rho} + \sum_{\rho} \frac{x^{\rho-1}}{1-\rho} \right) + \frac{1}{2} \log \frac{x+1}{x-1} - \frac{1}{x}.$$

The sum in brackets can be written as

$$\sum_{\rho} \frac{x^{\rho-1}}{\rho(1-\rho)}$$

which is an absolutely convergent series and is thus equal to

$$\sum_{\rho} \frac{x^{\rho-1}}{\rho(1-\rho)} = L(x) - \log x + \gamma + \sum_{\rho} \frac{x^{\rho-1}}{\rho} - \frac{1}{2} \log \frac{x+1}{x-1} + \frac{1}{x}.$$

The sum

$$\sum_{\rho} \frac{x^{\rho-1}}{\rho}$$

is equal to

$$\frac{x - \psi_0(x)}{x} - \frac{\log 2\pi}{x} - \frac{1}{2x} \log\left(1 - \frac{1}{x^2}\right)$$

and hence

$$\sum_{\rho} \frac{x^{\rho}}{\rho(1-\rho)} - \gamma x + \log 2\pi = 1 + x(L(x) - \log x) + x - \psi_0(x) - \frac{x}{2} \log \frac{x+1}{x-1} - \frac{1}{2} \log\left(1 - \frac{1}{x^2}\right).$$

As the right-hand side is an extended Baker period for algebraic x, this proves the following.

THEOREM 3.8. *For $x > 1$,*

$$S(x) := \sum_{\rho} \frac{x^{\rho}}{\rho(1-\rho)} - \gamma x + \log 2\pi$$

is an extended Baker period. In particular, assuming Schanuel's conjecture, the set

$$\left\{ \sum_{\rho} \frac{x^{\rho}}{\rho(1-\rho)} : x > 1, x \in \overline{\mathbb{Q}} \right\}$$

contains at most one algebraic number.

We remark that an expression similar to ours in the above theorem was also obtained by Ramaré [19] (however, sign in the sum over the zeros in his Lemma 2.2 should be negative).

§4. *Sums of general type involving the Riemann zeta function.* We now consider more general sums of the form

$$f(x) := \sum_{\rho} \frac{A(\rho)}{B(\rho)} x^{\rho},$$

where $A(t) \in \overline{\mathbb{Q}}[t]$ while $B(t) \in \mathbb{Q}[t]$ be polynomials and $x \in (0, \infty)$. We assume that $B(t)$ has simple rational roots with degree greater than that of $A(t)$. As before, the sum is defined following the convention of §3.

We shall need the following elementary lemma whose proof we omit (see [6, p. 137]).

LEMMA 4.1. *For $|z| < 1$, let*

$$f_u(z) = \sum_{n=1}^{\infty} \frac{z^n}{n+u}.$$

If $u = p/q$ is a rational number, then

$$f_u(z) = -z^{p/q} \sum_{m=0}^{q-1} \zeta_q^{-pm} \log(1 - \zeta_q^m z^{1/q}),$$

where $\zeta_q = \mathrm{e}^{2\pi \mathrm{i}/q}$.

We first consider the case when $x > 1$. For this, we shall further assume that $B(t)$ has simple rational roots lying in $\mathbb{Q} \backslash \{1, -2, -4, -6, \ldots\}$. We have the following theorem.

THEOREM 4.2. *Let $A(t)$ and $B(t)$ be as described above and let $\alpha_1, \ldots, \alpha_d$ be the roots of $B(t)$. For an algebraic number $x > 1$,*

$$g(x) := \sum_{\rho} \frac{A(\rho)}{B(\rho)} x^{\rho} + \sum_{i=1}^{d} \frac{A(\alpha_i)}{B'(\alpha_i)} \frac{\zeta'}{\zeta}(\alpha_i) x^{\alpha_i}$$

is an extended Baker period. Further for $\lambda_i := A(\alpha_i)/B'(\alpha_i)$,

- *if $\sum_{i=1}^{d} (\lambda_i/(1-\alpha_i)) \neq 0$, then $g(x)$ has at most one algebraic zero in $(1, \infty)$;*
- *if $\sum_{i=1}^{d} (\lambda_i/(1-\alpha_i)) = 0$ and $g(x) \neq 0$ for some algebraic $x > 1$, then at least one of the two numbers*

$$\sum_{\rho} \frac{A(\rho)}{B(\rho)} x^{\rho}, \qquad \sum_{i=1}^{d} \lambda_i \frac{\zeta'}{\zeta}(\alpha_i) x^{\alpha_i}$$

  is transcendental.

Proof. For any $x > 1, \alpha \in \mathbb{R}\backslash\{1, -2, -4, -6, \ldots\}$, we have the following expression

$$\psi_0(x, \alpha) = \frac{x}{1-\alpha} - x^\alpha \frac{\zeta'}{\zeta}(\alpha) - \sum_\rho \frac{x^\rho}{\rho - \alpha} + \sum_{n=1}^\infty \frac{x^{-2n}}{2n+\alpha},$$

where

$$\psi_0(x, \alpha) := \begin{cases} x^\alpha \sum_{n \leqslant x} \frac{\Lambda(n)}{n^\alpha} & \text{if } x \text{ is not a prime power;} \\ x^\alpha \sum_{n < x} \frac{\Lambda(n)}{n^\alpha} + \frac{1}{2}\Lambda(x) & \text{otherwise.} \end{cases}$$

This follows by modifying the explicit formula suitably. We now re-write this as

$$\sum_\rho \frac{x^\rho}{\rho - \alpha} + x^\alpha \frac{\zeta'}{\zeta}(\alpha) = \frac{x}{1-\alpha} - \psi_0(x, \alpha) + \sum_{n=1}^\infty \frac{x^{-2n}}{2n+\alpha}. \qquad (6)$$

Note that by Lemma 4.1, when α is a rational number and x is algebraic, the right-hand side of (6) is an extended Baker period. Now by using partial fractions, we can write

$$\frac{A(t)}{B(t)} = \sum_{i=1}^d \frac{\lambda_i}{t - \alpha_i}$$

with $\lambda_i := A(\alpha_i)/B'(\alpha_i)$. Thus the function

$$\sum_\rho \frac{A(\rho)}{B(\rho)} x^\rho + \sum_{i=1}^d \frac{A(\alpha_i)}{B'(\alpha_i)} \frac{\zeta'}{\zeta}(\alpha_i) x^{\alpha_i}$$

is equal to

$$x \sum_{i=1}^d \frac{\lambda_i}{1-\alpha_i} - \sum_{i=1}^d \lambda_i \psi_0(x, \alpha_i) + \sum_{i=1}^d \lambda_i \sum_{n=1}^\infty \frac{x^{-2n}}{2n+\alpha_i}.$$

which is an extended Baker period if x is algebraic. The second part of the theorem is again a consequence of Baker's theorem. □

Finally, when $x \in (0, 1)$ and $\alpha \in \mathbb{R}\backslash\{0, 1, 3, 5, 7, \ldots\}$, we have

$$T(x, \alpha) = \sum_\rho \frac{x^\rho}{\rho - \alpha} + \frac{1}{\alpha} - x^\alpha \frac{\zeta'}{\zeta}(1-\alpha) + \sum_{n=1}^\infty \frac{x^{2n+1}}{2n+1-\alpha},$$

where

$$T(x, \alpha) := \begin{cases} x^\alpha \sum_{n \leqslant 1/x} \frac{\Lambda(n)}{n^{1-\alpha}} & \text{if } 1/x \text{ is not a prime power,} \\ x^\alpha \sum_{n < 1/x} \frac{\Lambda(n)}{n^{1-\alpha}} + \frac{x}{2}\Lambda(1/x) & \text{otherwise.} \end{cases}$$

Hence we have

$$\sum_{\rho} \frac{x^{\rho}}{\rho - \alpha} - x^{\alpha} \frac{\zeta'}{\zeta}(1-\alpha) = T(x,\alpha) - \frac{1}{\alpha} - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1-\alpha},$$

where the right-hand side is an extended Baker period when x is algebraic and α is rational. Thus we have the following theorem for $x < 1$.

THEOREM 4.3. *Let $A(t)$ and $B(t)$ be as before and let $\alpha_1, \ldots, \alpha_d$ be the roots of $B(t)$ all of which lie in $\mathbb{Q}\backslash\{0, 1, 3, 5, 7, \ldots\}$. For an algebraic $x \in (0, 1)$, the number*

$$h(x) := \sum_{\rho} \frac{A(\rho)}{B(\rho)} x^{\rho} - \sum_{i=1}^{d} \frac{A(\alpha_i)}{B'(\alpha_i)} \frac{\zeta'}{\zeta}(1-\alpha_i) x^{\alpha_i}$$

is an extended Baker period. Further, the set

$$\{h(x) \mid x \in (0,1) \cap \overline{\mathbb{Q}}\}$$

can have at most one algebraic number.

Proof. As before, using partial fractions, we can deduce that

$$h(x) = \sum_{i=1}^{d} \lambda_i T(x,\alpha_i) - \sum_{i=1}^{d} \frac{\lambda_i}{\alpha_i} - \sum_{i=1}^{d} \lambda_i \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1-\alpha_i},$$

where $\lambda_i := A(\alpha_i)/B'(\alpha_i)$. When x is algebraic, $h(x)$ is an extended Baker period. Finally the second part of the theorem follows by noting that for x, y algebraic, $h(x) - h(y)$ is a Baker period. □

§5. *The Selberg class.* Selberg [20] defined a large class $\mathcal{S}$ of Dirichlet series admitting analytic continuation and functional equation. It is likely that this class includes the universe of automorphic L-functions, though this has not yet been proven. The class $\mathcal{S}$ is defined as follows.

(1) Each $F \in \mathcal{S}$ is a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_F(n) n^{-s},$$

absolutely convergent for $\Re(s) > 1$.

(2) There exists an integer $m \geqslant 0$ such that $(s-1)^m F(s)$ is an entire function of finite order. Let m_F denote the least value of such m.

(3) For each $F \in \mathcal{S}$, there exist numbers $Q_F > 0$ and $r \geqslant 0$, and numbers $\lambda_j > 0$ and μ_j with $\Re(\mu_j) \geqslant 0$, such that

$$\xi_F(s) := Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

satisfies the functional equation

$$\xi_F(s) = w\overline{\xi_F}(1-s),$$

with w a complex number of absolute value 1. Here $\overline{\xi_F}(s) = \overline{\xi_F(\overline{s})}$ and an empty product equals 1.

(4) The Dirichlet coefficients $a_F(n)$ satisfy $a_F(n) \ll n^{\epsilon}$ for every $\epsilon > 0$.

(5) $\log F(s)$ can be written as a Dirichlet series

$$\sum_{n=1}^{\infty} b_F(n) n^{-s},$$

where $b_F(n)$ is zero unless n is a prime power and $b_F(n) \ll n^{\theta}$ for some $\theta < 1/2$.

There are several celebrated conjectures related to this class and we refer the reader to [20] and [13] for further details. Because of the Legendre duplication formula for the Γ-function, it is easy to see that the functional equation is not unique for an arbitrary element F in $\mathcal{S}$. However, the invariants

$$d_F = 2\sum_{j=1}^{r} \lambda_j, \qquad q_F = (2\pi)^{d_F} Q_F^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}, \qquad \theta_F = 2\Im\left(\sum_{j=1}^{r} (\mu_j - 1/2)\right),$$

are well defined and called, the *degree*, the *conductor* and *shift*, respectively. One conjectures that d_F and q_F are positive integers. Recently, some impressive work [11] has appeared that shows that $0 < d_F < 1$ and $1 < d_F < 2$ are impossible.

We now derive a general formula for an element in the Selberg class. It is convenient to write

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \Lambda_F(n) n^{-s}, \qquad \Lambda_F(n) = b_F(n) \log n.$$

For $x > 1$, let us introduce the notation

$$\psi_0(x, F, \alpha) := \begin{cases} x^{\alpha} \displaystyle\sum_{n \leqslant x} \frac{\Lambda_F(n)}{n^{\alpha}} & \text{if } x \text{ is not a prime power,} \\ x^{\alpha} \displaystyle\sum_{n < x} \frac{\Lambda_F(n)}{n^{\alpha}} + \frac{1}{2}\Lambda_F(x) & \text{otherwise.} \end{cases}$$

Let α be a complex number not equal to any of the poles and zeros of $F(s)$. Again the explicit formula for F (see formula (12) of [15], for instance) yields that

$$\psi_0(x, F, \alpha) = -x^\alpha \frac{F'}{F}(\alpha) + \frac{m_F x}{1-\alpha} - \sum_\rho \frac{x^\rho}{\rho - \alpha} + \sum_{j=1}^{r} \sum_{n=0}^{\infty} \frac{x^{-((n+\mu_j)/\lambda_j)}}{(n+\mu_j)/\lambda_j + \alpha} - \frac{m_F}{\alpha},$$

where ρ runs over the non-trivial zeros of F in the sector $0 \leqslant \Re(s) \leqslant 1$. Recalling the following series introduced earlier,

$$f_u(z) = \sum_{n=1}^{\infty} \frac{z^n}{n+u},$$

we have

$$\sum_{n=1}^{\infty} \frac{x^{-((n+\mu_j)/\lambda_j)}}{(n+\mu_j)/\lambda_j + \alpha} = x^{-\mu_j/\lambda_j} \lambda_j f_{\mu_j + \alpha\lambda_j}(x^{-1/\lambda_j}),$$

and hence

$$x^\alpha \frac{F'(\alpha)}{F(\alpha)} + \sum_\rho \frac{x^\rho}{\rho - \alpha}$$

is equal to

$$\frac{m_F x}{1-\alpha} - \psi_0(x, F, \alpha) + \sum_{j=1}^{r} x^{-\mu_j/\lambda_j} \lambda_j f_{\mu_j + \alpha\lambda_j}(x^{-1/\lambda_j}) + \sum_{j=1}^{r} \frac{x^{-(\mu_j/\lambda_j)}}{\mu_j/\lambda_j + \alpha} - \frac{m_F}{\alpha}.$$

Again let $A(t) \in \overline{\mathbb{Q}}[t]$ and $B(t) \in \mathbb{Q}[t]$ be polynomials such that $B(t)$ has simple rational roots not equal to any of the poles and zeros of $F(s)$ and degree of $B(t)$ is strictly greater than the degree of $A(t)$. Then as before,

$$\frac{A(t)}{B(t)} = \sum_{i=1}^{d} \frac{\beta_i}{t - \alpha_i}$$

with $\beta_i := A(\alpha_i)/B'(\alpha_i)$, and hence we have

$$\sum_\rho \frac{A(\rho)}{B(\rho)} x^\rho + \sum_{i=1}^{d} \frac{A(\alpha_i)}{B'(\alpha_i)} \frac{F'}{F}(\alpha_i) x^{\alpha_i}$$

is equal to

$$m_F x \sum_{i=1}^{d} \frac{\beta_i}{1-\alpha_i} - \sum_{i=1}^{d} \beta_i \psi_0(x, F, \alpha_i)$$
$$+ \sum_{i=1}^{d} \beta_i \left\{ \sum_{j=1}^{r} x^{-\mu_j/\lambda_j} \lambda_j f_{\mu_j+\alpha_i\lambda_j}(x^{-1/\lambda_j}) \right\}$$
$$+ \sum_{i=1}^{d} \beta_i \left\{ \sum_{j=1}^{r} \frac{x^{-(\mu_j/\lambda_j)}}{\mu_j/\lambda_j + \alpha_i} - \frac{m_F}{\alpha_i} \right\}.$$

We will say two functions $F, G \in \mathcal{S}$ are of the same *Hodge type* if they admit a functional equation with the same λ_j and the μ_j. In such a situation, we can prove the following.

THEOREM 5.1. *Consider the set of elements $F \in \mathcal{S}$ with a fixed Hodge type such that*

$$\Delta_F := \sum_{\rho} \frac{A(\rho)}{B(\rho)} x^{\rho} + \sum_{i=1}^{d} \frac{A(\alpha_i)}{B'(\alpha_i)} \frac{F'}{F}(\alpha_i) x^{\alpha_i}$$

is algebraic for algebraic $x > 1$. Further assume that $\beta_i = A(\alpha_i)/B'(\alpha_i)$ are real with same sign for $1 \leqslant i \leqslant d$. If there are two elements F, G in this set, then for any prime p with $\sqrt{x} < p \leqslant x$, we have

$$b_F(p) = b_G(p).$$

In particular, if $2 \leqslant x < 3$, then $b_F(2) = b_G(2)$.

Proof. For any such elements F and G, the above discussion along with Baker's theorem would necessarily imply that

$$\sum_{i=1}^{d} \beta_i \psi_0(x, F, \alpha_i) = \sum_{i=1}^{d} \beta_i \psi_0(x, G, \alpha_i).$$

The theorem then follows as logarithms of primes are linearly independent over $\mathbb{Q}$ and hence over $\overline{\mathbb{Q}}$ by Baker's theorem. □

§6. *The arithmetic Selberg class $\mathcal{A}$.* We now focus our attention on a subclass $\mathcal{A}$ of $\mathcal{S}$, which we call the *arithmetic Selberg class*. The class $\mathcal{A}$ is defined by the following axioms.

(1) Each $F \in \mathcal{A}$ is a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_F(n) n^{-s},$$

absolutely convergent for $\Re(s) > 1$.

(2) There exists an integer $m \geqslant 0$ such that $(s-1)^m F(s)$ is an entire function of finite order. As before, m_F is the smallest such m.
(3) For each $F \in \mathcal{A}$, there exist numbers Q_F and r, and *rational numbers* $\lambda_j > 0$ and $\mu_j \geqslant 0$ such that

$$\xi_F(s) := Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

satisfies the functional equation

$$\xi_F(1-s) = w\overline{\xi_F}(1-s),$$

with w a complex number of absolute value 1. Moreover, the conductor q_F is assumed to be a *natural number*.
(4) $\log F(s)$ can be written as a Dirichlet series

$$\sum_{n=1}^{\infty} b_F(n) n^{-s},$$

with $b_F(n)$ *algebraic* satisfying $b_F(n) = 0$ if n is not a power of p, with p prime.

Technically speaking, $\mathcal{A}$ is not a subclass of $\mathcal{S}$ since the reader will note that the Ramanujan estimate for the coefficients $a_F(n)$ which appears in the definition of the Selberg class $\mathcal{S}$, is not assumed in the above definition since it is not essential for the nature of the theorems we will derive. We also do not assume any estimate for $b_F(n)$. It is also easy to see that the algebraicity of $b_F(n)$ implies the algebraicity of $a_F(n)$.

Most of the zeta functions that arise in number theory (such as the Artin L-functions and zeta functions attached to algebraic varieties) either belong to $\mathcal{A}$ or are expected to belong to $\mathcal{A}$.

As earlier, for $F \in \mathcal{A}$ and for $x > 1$,

$$x^{\alpha} \frac{F'}{F}(\alpha) + \sum_{\rho} \frac{x^{\rho}}{\rho - \alpha}$$

is equal to

$$\frac{m_F x}{1-\alpha} + \sum_{j=1}^{r} \frac{x^{-(\mu_j/\lambda_j)}}{\mu_j/\lambda_j + \alpha} - \frac{m_F}{\alpha} - \psi_0(x, F, \alpha) + \sum_{j=1}^{r} x^{-\mu_j/\lambda_j} \lambda_j f_{\mu_j + \alpha\lambda_j}(x^{-1/\lambda_j}).$$

From this formula, we see that for x algebraic, the right-hand side is an extended Baker period provided α, the μ_j and the λ_j are all rational numbers.

THEOREM 6.1. *Let $F \in \mathcal{A}$. Let $A(t)$, $B(t)$ be polynomials as before and $B(t)$ of degree d with simple rational roots $\alpha_1, \ldots, \alpha_d$ not equal to the zeros and poles of F. For $x > 1$ and algebraic, we have that*

$$g(x) := \sum_{\rho} \frac{A(\rho)}{B(\rho)} x^{\rho} + \sum_{i=1}^{d} \beta_i \frac{F'}{F}(\alpha_i) x^{\alpha_i},$$

where $\beta_i := A(\alpha_i)/B'(\alpha_i)$, is an extended Baker period. Further suppose that $\mu_j = 0$ for all j. Then:

- *if $m_F \sum_{i=1}^{d} (\beta_i/(1-\alpha_i)) \neq 0$, then $g(x)$ has at most one algebraic zero in $(1, \infty)$;*
- *if $m_F \sum_{i=1}^{d} (\beta_i/(1-\alpha_i)) = 0$, then the set*

$$\{g(x) \mid x \in (1, \infty) \cap \overline{\mathbb{Q}}\}$$

has at most one algebraic number.

Proof. The proof follows from the preceding discussion and appealing to Baker's theorem. □

§7. *The case $0 < x < 1$ for the Selberg class.* When $0 < x < 1$, recall that one has the following expression as indicated by Ingham:

$$\sum_{n \leqslant 1/x}{}' \frac{\Lambda(n)}{n} = -\log x - \gamma + \sum_{\rho} \frac{x^{\rho}}{\rho} + \frac{1}{2} \log \frac{1+x}{1-x} - x,$$

where γ denotes the Euler's constant. As noted earlier, this is deduced by considering the following integral

$$\frac{1}{2\pi \mathrm{i}} \int_{3-\mathrm{i}\infty}^{3+\mathrm{i}\infty} \frac{x^{1-s}}{1-s} \frac{\zeta'}{\zeta}(s)\, ds.$$

A similar argument can be applied to an arbitrary element F in the Selberg class. Let us write

$$-\frac{F'}{F}(s) = \frac{m_F}{s-1} - \gamma_F + O(s-1).$$

Then, for $x \in (0, 1)$ such that $1/x$ is not a prime power, we have

$$\sum_{n \leqslant 1/x} \frac{\Lambda_F(n)}{n} = -m_F \log x - \gamma_F + \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{j=1}^{r} \left\{ \lambda_j x^{1+\mu_j/\lambda_j} f_{\lambda_j+\mu_j}(x^{1/\lambda_j}) + \frac{\lambda_j x^{1+\mu_j/\lambda_j}}{\lambda_j + \mu_j} \right\},$$

where ρ runs over the non-trivial zeros of $\overline{F}(s)$. Finally, when $x \in (0, 1)$ and $\alpha \neq 0$ and also not equal to poles and zeros of F, we have

$$T(x, F, \alpha) = \sum_{\rho} \frac{x^{\rho}}{\rho - \alpha} + \frac{m_F}{\alpha} + \frac{m_F x}{1 - \alpha} - x^{\alpha} \frac{F'}{F}(1 - \alpha) + \sum_{j=1}^{r} \lambda_j \frac{x^{1+(\mu_j/\lambda_j)}}{\mu_j + \lambda_j - \alpha\lambda_j} + \sum_{j=1}^{r} \lambda_j x^{1+\mu_j/\lambda_j} f_{\mu_j+\lambda_j-\alpha\lambda_j}(x^{1/\lambda_j}),$$

where ρ runs over the non-trivial zeros of $\overline{F}(s)$. Here

$$T(x, F, \alpha) := \begin{cases} x^{\alpha} \displaystyle\sum_{n \leqslant 1/x} \frac{\Lambda_F(n)}{n^{1-\alpha}} & \text{if } 1/x \text{ is not a prime power,} \\ x^{\alpha} \displaystyle\sum_{n < 1/x} \frac{\Lambda_F(n)}{n^{1-\alpha}} + \frac{x}{2}\Lambda_F(1/x) & \text{otherwise.} \end{cases}$$

Hence we have

$$\sum_{\rho} \frac{x^{\rho}}{\rho - \alpha} - x^{\alpha} \frac{F'}{F}(1 - \alpha) = T(x, F, \alpha) - \frac{m_F}{\alpha} - \frac{m_F x}{1 - \alpha} - \sum_{j=1}^{r} \frac{\lambda_j x^{1+(\mu_j/\lambda_j)}}{\mu_j + \lambda_j - \alpha\lambda_j} - \sum_{j=1}^{r} \lambda_j x^{1+\mu_j/\lambda_j} f_{\mu_j+\lambda_j-\alpha\lambda_j}(x^{1/\lambda_j}).$$

Recalling that two functions $F, G \in \mathcal{S}$ are of the same *Hodge type* if they admit a functional equation with the same λ_j and the μ_j, we have the following theorem.

THEOREM 7.1. *Let $A(t)$, $B(t)$ be as in Theorem 7.2. Consider the set of elements $F \in \mathcal{S}$ with a fixed Hodge type such that*

$$h(x) := \sum_{\rho} \frac{A(\rho)}{B(\rho)} x^{\rho} - \sum_{i=1}^{d} \frac{A(\alpha_i)}{B'(\alpha_i)} \frac{F'}{F}(1 - \alpha_i) x^{\alpha_i}$$

is algebraic for algebraic $x < 1$. Further assume that $\beta_i = A(\alpha_i)/B'(\alpha_i)$ are real with same sign for $1 \leqslant i \leqslant d$. If there are two elements F, G in this set, then for any prime p with $1/\sqrt{x} < p \leqslant 1/x$, we have

$$b_F(p) = b_G(p).$$

Proof. The proof follows by arguing along the line of the proof of Theorem 5.1. □

Finally, when F is in the arithmetic Selberg class, we have the following theorem whose proof is analogous to that of Theorem 6.1.

THEOREM 7.2. *Let $F \in \mathcal{A}$. Let $A(t)$ and $B(t)$ be as before and let $\alpha_1, \ldots, \alpha_d$ be the roots of $B(t)$ which are all rational, non-zero, simple and not equal to the zeros and poles of F. Then for an algebraic $x \in (0, 1)$, the number*

$$h(x) := \sum_{\rho} \frac{A(\rho)}{B(\rho)} x^{\rho} - \sum_{i=1}^{d} \frac{A(\alpha_i)}{B'(\alpha_i)} \frac{F'}{F}(1-\alpha_i) x^{\alpha_i}$$

is an extended Baker period. Here ρ runs over the non-trivial zeros of $\overline{F}(s)$. Further suppose that $\mu_j = 0$ for all j. Then:

- *if $m_F \sum_{i=1}^{d} (\beta_i/(1-\alpha_i)) \neq 0$, then $h(x)$ has at most one algebraic zero in $(0, 1)$;*
- *if $m_F \sum_{i=1}^{d} (\beta_i/(1-\alpha_i)) = 0$, then the set*

$$\{h(x) \mid x \in (0, 1) \cap \overline{\mathbb{Q}}\}$$

has at most one algebraic number.

§8. *Concluding remarks.* It is yet unclear what role (if any) transcendental number theory plays in our journey towards the grand Riemann hypothesis. The generalized Li criterion as well as many of the theorems of this paper suggest that there may be a link. If so, this paper represents a humble beginning towards our lofty goal.

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