

A NOTE ON SPECIAL VALUES OF L -FUNCTIONS

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ABSTRACT. In this paper, we link the nature of special values of certain Dirichlet L -functions to those of multiple gamma values.

1. INTRODUCTION

Special values of L -functions are one of the deepest mysteries in mathematics, and it is fair to say that our knowledge about this world is still in its infancy. In this note, we investigate these values in terms of multiple gamma functions.

The multiple Γ -functions were introduced around 1900 by Barnes and Glaisher and have now come into prominence due to the works of Shintani [19], Sarnak [18], Keating and Snaith [10], etc., in relation to the analytic study of the Riemann zeta function. The goal of this work is to relate the algebraic nature of special values of these mysterious functions to that of $\zeta(3)$ and Catalan's constant. Our investigation is based upon the work of Nesterenko [14] in transcendence theory and the recent work of Adamchik [1].

We first define the multiple Γ -functions $\Gamma_m(x)$. They are natural generalizations of the classical Γ -function. As a real function, the multiple gamma function $\Gamma_m(x)$ for positive x and $m \geq 0$ is defined as follows:

$$\begin{aligned}\Gamma_0(x) &= \frac{1}{x}, \quad \Gamma_m(1) = 1, \quad \Gamma_{m+1}(x+1) = \frac{\Gamma_{m+1}(x)}{\Gamma_m(x)}, \\ \frac{1}{\Gamma_m(x)} &\text{ is } C^\infty \text{ on } \mathbb{R}, \\ (-1)^{m+1} \frac{d^{m+1}}{dx^{m+1}} \log \Gamma_m(x) &\geq 0 \text{ for } x > 0.\end{aligned}$$

The existence and uniqueness of $\Gamma_m(x)$ follows from the works of Dufresnoy and Pisot (see [8] for details). This can be regarded as a generalisation of the Bohr-Mollerup theorem, which asserts that $\Gamma_1(x)$ is equal to the classical gamma function.

Here, we have the following theorems:

Theorem 1.1. *At least one of the following is true:*

- (i) *The number $(\zeta(3)/\pi^3)^2$ is irrational.*
- (ii) *$\Gamma_3(1/2)\Gamma_2(1/2)^{-1}$ is transcendental.*

Let G be Catalan's constant defined as

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

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Unlike $\zeta(3)$, its irrationality is yet to be established (we refer to the work of Rivoal and Zudilin [17] in relation to this constant). We have the following theorems:

Theorem 1.2. *At least one of the following is true:*

- (i) G/π^2 is irrational, where G is Catalan’s constant.
- (ii) $\Gamma_2(1/4)\Gamma_2(3/4)^{-1}$ is transcendental.

Theorem 1.3. *At least one of the following is true:*

- (i) $L(2, \chi_3)/\pi^2$ is irrational, where χ_3 is the odd character modulo 3.
- (ii) $\Gamma_2(1/3)\Gamma_2(2/3)^{-1}$ is transcendental.

Further, we investigate from the viewpoint of Schanuel’s conjecture (see §2), which is about the algebraic independence of exponential values. We show that the conjecture of Schanuel can remarkably refine Theorem 1.1 derived above. For instance, we have

Theorem 1.4. *Assume that Schanuel’s conjecture is true. Then at least one of the following is true:*

- (i) $\zeta(3)$ and π are algebraically independent.
- (ii) $\Gamma_3(1/2)\Gamma_2(1/2)^{-1}$ is transcendental.

Motivated by this and the works of Nesterenko, we propose a question which is an elliptic-exponential generalisation of Schanuel’s conjecture (see §7). Our question seems to follow from a more general conjecture of Bertolin [4] and is related to conjectures of Grothendieck (see [7], [2] as well as the remarks at the end of our paper).

Finally, for Dirichlet characters χ modulo some integer $q > 1$ and integers $k > 1$, the values $L(k, \chi)$ lie in the field $\overline{\mathbb{Q}}(\pi)$ when k and χ have the same parity. However, when k is fixed and χ and k have different parity, we expect these $\varphi(q)/2$ numbers to be unrelated and generate new transcendental numbers. In this connection, it is worthwhile to mention the recent paper of Lutes and Papanikolas [12], where analogous questions for the Goss L -functions are considered and partial results have been obtained (see Theorem 1.2, for instance).

In the following theorem, we note that these unknown numbers can be generated by derivatives of Riemann zeta values at rational arguments. More precisely, we have

Theorem 1.5. *For integers $k, q > 1$, let S be the set*

$$S := \left\{ \zeta^{(j)}(a/q) : 0 \leq j \leq k, 1 \leq a \leq q - 1 \right\} \cup \{\pi\}.$$

Also let f be an algebraic valued periodic function with period q . Then one has $L(k, f) \in \overline{\mathbb{Q}}(S)$. In particular, for any Dirichlet character χ modulo q , we have

$$L(k, \chi) \in \overline{\mathbb{Q}}(S).$$

In the special case of $q = 2$, we have the following interesting corollary:

Corollary 1.6. *Let S be the set*

$$S := \left\{ \zeta^{(j)}(1/2) : j = 0, 1, 2, 3, \dots \right\} \cup \{\pi\}.$$

Then for all $k > 1$, we have

$$\zeta(k) \in \overline{\mathbb{Q}}(S).$$

2. SOME PREREQUISITES

Let ψ be the digamma function which is the logarithmic derivative of the classical gamma function. We have for $z \neq 0, -1, -2, \dots$,

$$-\psi(z) = \gamma + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right).$$

Here γ is Euler's constant. For $i \geq 1$, we have the following i -th derivatives of the digamma function which are referred to as poly-gamma functions:

$$(1) \quad \psi_i(z) = (-1)^{i-1} i! \sum_{n=0}^{\infty} \frac{1}{(n+z)^{i+1}}.$$

Next, we define the multiple Γ -functions $\Gamma_m(z)$ for complex z . We have already defined $\Gamma_m(x)$ for real positive x . As in the case of a classical gamma function, $\frac{1}{\Gamma_m(x)}$ can be extended to an entire function of order m with the Hadamard factorization

$$\frac{1}{\Gamma_m(z+1)} = e^{P_m(z)} \prod_{n \geq 1} \left(\left(1 + \frac{z}{n} \right) e^{\left(-\frac{z}{n} + \frac{z^2}{2n^2} - \dots + \frac{(-z)^m}{mn^m} \right)} \right)^{\binom{n+m-2}{m-1}}.$$

Here $P_m(z)$ is a polynomial of degree m (see [21]). In particular, Γ_2 and Γ_3 are given by

$$\begin{aligned} \frac{1}{\Gamma_2(z+1)} &= e^{P_2(z)} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n} \right)^n e^{-z + \frac{z^2}{2n}} \right\}, \\ \frac{1}{\Gamma_3(z+1)} &= e^{P_3(z)} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n} \right)^{\frac{n(n+1)}{2}} e^{\left(-z + \frac{z^2}{2n} - \frac{z^3}{3n^2} \right) \frac{n+1}{2}} \right\}, \end{aligned}$$

where

$$\begin{aligned} P_2(z) &= -\frac{1}{2}[(1 + \gamma)z^2 + z] + \frac{z}{2} \log 2\pi, \\ P_3(z) &= \left(\zeta'(-1) - \frac{\log 2\pi}{4} + \frac{7}{24} \right) z + \left(\frac{\gamma + \log 2\pi}{4} + \frac{1}{8} \right) z^2 \\ &\quad - \left(\frac{\gamma}{6} + \frac{\pi^2}{36} + \frac{1}{4} \right) z^3. \end{aligned}$$

They were first studied by Barnes [3] and often referred to as Barnes gamma functions. For basic properties of these functions, we refer the reader to [1], [8], [9], [20], [21].

Recently, Adamchik has expressed derivatives of Hurwitz zeta function in terms of multiple gamma function $\Gamma_m(z)$ as follows (see Proposition 3 of [1]):

$$\zeta'(-m, z) - \zeta'(-m) = (-1)^m \sum_{n=0}^m n! Q_{n,m}(z) \log \Gamma_{n+1}(z) \text{ for } \Re(z) > 0,$$

where

$$Q_{n,m}(z) = (-1)^n \sum_{j=n}^m (1-z)^{m-j} \binom{m}{j} \left\{ \begin{matrix} j \\ n \end{matrix} \right\}$$

is a polynomial with rational coefficients. Here $\left\{ \begin{smallmatrix} j \\ n \end{smallmatrix} \right\}$ are the Stirling numbers of the second kind defined by

$$\left\{ \begin{smallmatrix} j \\ n \end{smallmatrix} \right\} := \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^j.$$

When $n = 0$, this is 1 for $j = 0$ and zero for $j > 0$. In particular, for $m = 2, 1$ and $\Re(z) > 0$, one has

$$(2) \quad \zeta'(-2, z) - \zeta'(-2) = 2 \log \Gamma_3(z) - (3 - 2z) \log \Gamma_2(z) + (1 - z)^2 \log \Gamma(z),$$

$$(3) \quad \zeta'(-1, z) - \zeta'(-1) = \log \Gamma_2(z) + (z - 1) \log \Gamma(z).$$

We recall that for a real number $0 < x \leq 1$ and $s \in \mathbb{C}$ with $\Re(s) > 1$, the Hurwitz zeta function $\zeta(s, x)$ is defined as

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.$$

It extends meromorphically to the whole complex plane with a simple pole at $s = 1$ with residue 1. Let us note that $\zeta(s, 1) = \zeta(s)$ and that

$$(q^s - 1)\zeta(s) = \sum_{a=1}^{q-1} \zeta(s, a/q).$$

One of the crucial ingredients in our proofs is the following theorem due to Nesterenko [14] (see also page 6 of [15]).

Theorem 2.1 (Nesterenko). *For any imaginary quadratic field with discriminant $-D$ and character ϵ , the numbers*

$$\pi, e^{\pi\sqrt{D}}, \prod_{a=1}^{D-1} \Gamma(a/D)^{\epsilon(a)}$$

are algebraically independent. Thus, in particular the three numbers $\Gamma(1/4), \pi$ and e^{π} are algebraically independent and so are the three numbers $\Gamma(1/3), \pi$ and $e^{\pi\sqrt{3}}$.

We end the section by recalling a conjecture of Schanuel [11].

Schanuel’s conjecture. *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be complex numbers which are linearly independent over \mathbb{Q} . Then the transcendence degree of the field*

$$\mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_n})$$

over \mathbb{Q} is at least n .

This includes almost all known results on transcendence as well as all reasonable conjectures on the values of the exponential function. For instance, it implies the algebraic independence of e and π .

3. PROOF OF THEOREM 1.1

For all $s \in \mathbb{C}$, we have

$$\zeta(1 - s) = 2(2\pi)^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s).$$

Thus,

$$-\zeta'(-2) = \frac{\zeta(3)}{4\pi^2}.$$

Further,

$$\zeta(s, 1/2) = (2^s - 1)\zeta(s),$$

and hence

$$\zeta'(s, 1/2) = (2^s - 1)\zeta'(s) + 2^s(\log 2)\zeta(s).$$

Evaluating at $s = -2$, we get

$$\zeta'(-2, 1/2) = -\frac{3}{4}\zeta'(-2).$$

Now by evaluating (2) at $z = 1/2$, we get

$$\zeta'(-2, 1/2) - \zeta'(-2) = 2 \log \Gamma_3(1/2) - 2 \log \Gamma_2(1/2) + \frac{1}{4} \log \Gamma(1/2).$$

Hence we have

$$-\frac{7}{4}\zeta'(-2) = 2 \log \left(\frac{\Gamma_3(1/2)\pi^{1/16}}{\Gamma_2(1/2)} \right).$$

Thus

$$(4) \quad \frac{\zeta(3)}{\pi^3} = \frac{32}{7\pi} \log \left(\frac{\Gamma_3(1/2)\pi^{1/16}}{\Gamma_2(1/2)} \right).$$

Now suppose that

$$\frac{\zeta(3)}{\pi^3} = A\sqrt{d}, \quad \text{for some } A \in \mathbb{Q} \text{ and } d \in \mathbb{N},$$

where d can be assumed to be square free. Then

$$\frac{\Gamma_3(1/2)\pi^{1/16}}{\Gamma_2(1/2)} = e^{\pi r \sqrt{d}},$$

where r is a rational number. Since by Nesterenko's theorem, π and $e^{\pi\sqrt{d}}$ are algebraically independent, we get that

$$\frac{\Gamma_3(1/2)}{\Gamma_2(1/2)}$$

is necessarily transcendental. This proves the theorem.

4. PROOFS OF THEOREM 1.2 AND THEOREM 1.3

Let χ be a primitive Dirichlet character modulo q . Then for all $s \in \mathbb{C}$, we have

$$L(1 - s, \chi) = \frac{q^{s-1}\Gamma(s)}{(2\pi)^s} \{e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\} G(1, \chi) L(s, \bar{\chi}),$$

where

$$G(1, \chi) = \sum_{a=1}^q \chi(a) e^{2\pi ia/q}.$$

Hence for the odd primitive Dirichlet character χ_4 modulo 4, one has

$$(5) \quad L(1 - s, \chi_4) = \left(\frac{2}{\pi}\right)^s \sin \frac{\pi s}{2} \Gamma(s) L(s, \chi_4).$$

Differentiating the above expression and then evaluating at $s = 2$, we get

$$L'(-1, \chi_4) = \frac{2}{\pi} G,$$

where G is Catalan’s constant. On the other hand, we know

$$L(s, \chi_4) = 4^{-s} \sum_{a=1}^4 \chi_4(a) \zeta(s, a/4).$$

Differentiating, we have

$$\begin{aligned} L'(s, \chi_4) &= \frac{1}{4^s} \sum_{a=1}^4 \chi_4(a) \zeta'(s, a/4) - \frac{\log 4}{4^s} \sum_{a=1}^4 \chi_4(a) \zeta(s, a/4) \\ &= \frac{1}{4^s} \sum_{a=1}^4 \chi_4(a) \zeta'(s, a/4) - (\log 4)L(s, \chi_4). \end{aligned}$$

Thus

$$L'(-1, \chi_4) = 4 \sum_{a=1}^4 \chi_4(a) \zeta'(-1, a/4)$$

as $L(-1, \chi_4) = 0$ by substituting $s = 2$ in (5). Using equation (3), we have

$$L'(-1, \chi_4) = 4 \sum_{a=1}^4 \chi_4(a) \left(\frac{a}{4} - 1\right) \log \Gamma(a/4) + 4 \sum_{a=1}^4 \chi_4(a) \log \Gamma_2(a/4).$$

Since $\chi_4(1) = 1$ and $\chi_4(3) = -1$, we have

$$\begin{aligned} \frac{2}{\pi}G &= L'(-1, \chi_4) \\ &= -3 \log \Gamma(1/4) + \log \Gamma(3/4) + 4 [\log \Gamma_2(1/4) - \log \Gamma_2(3/4)] \\ &= \log \frac{\Gamma(3/4)}{\Gamma(1/4)^3} + 4 \log \frac{\Gamma_2(1/4)}{\Gamma_2(3/4)}. \end{aligned}$$

Now suppose that

$$\frac{G}{\pi^2} = r,$$

where r is a rational number. This implies that

$$\frac{\Gamma(3/4)\Gamma_2(1/4)^4}{\Gamma(1/4)^3\Gamma_2(3/4)^4} = e^{2\pi r}.$$

Since

$$\Gamma(1/4)\Gamma(3/4) = \sqrt{2}\pi,$$

we have

$$\left[\frac{\Gamma_2(1/4)}{\Gamma_2(3/4)}\right]^4 = \frac{\Gamma(1/4)^4 e^{2\pi r}}{\sqrt{2}\pi}.$$

By Nesterenko’s theorem, the three numbers $\Gamma(1/4), \pi$ and e^π are algebraically independent. Thus the right hand side is necessarily transcendental, and hence we have the theorem.

Remark 4.1. When $q = 3$, we have the unique odd character χ_3 . Nothing is known about the nature of the number $L(2, \chi_3)$. Arguing exactly as above, we can prove Theorem 1.3, which sheds some light on this question.

5. PROOF OF THEOREM 1.4

Suppose that $\zeta(3)$ and π are algebraically dependent. Hence by (4), we have that

$$\log \left(\frac{\Gamma_3(1/2)}{\Gamma_2(1/2)} \right) + \frac{1}{16} \log \pi$$

and π are algebraically dependent. Now suppose that

$$\alpha = \frac{\Gamma_3(1/2)}{\Gamma_2(1/2)}$$

is an algebraic number. Then by Schanuel’s conjecture, the transcendence degree of the field

$$\mathbb{Q}(\log \alpha, i\pi, \log \pi, \alpha, -1, \pi)$$

is at least 3. But this contradicts the algebraic dependence of the numbers $\log(\Gamma_3(1/2)\Gamma_2(1/2)^{-1}) + \frac{1}{16} \log \pi$ and π . Hence the theorem.

6. PROOF OF THEOREM 1.5

We have

$$L(k, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^k} = q^{-k} \sum_{a=1}^q f(a)\zeta(k, a/q).$$

Consequently, using equation (1), we have

$$(6) \quad L(k, f) = \frac{(-1)^k}{q^k(k-1)!} \sum_{a=1}^q f(a)\psi_{k-1}(a/q).$$

Now for all $s \in \mathbb{C}$, we have

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s).$$

Taking the logarithmic derivative of the above, we have

$$\psi(s) = \log 2\pi - \frac{\zeta'}{\zeta}(1-s) - \frac{\zeta'}{\zeta}(s) + \frac{\pi}{2} \tan \frac{\pi s}{2}.$$

Differentiating the above expression $k-1$ times, we conclude that

$$\psi_{k-1}(a/q) \in \overline{\mathbb{Q}}(S).$$

Using the identity given by (6) and the above two identities, we conclude that

$$L(k, f) \in \overline{\mathbb{Q}}(S).$$

7. CONCLUDING REMARKS

The conjecture of Schanuel is about the algebraic independence of the values of the exponential function. Nesterenko proved the following general result (see [14], Chapter 3, Corollary 1.6).

Proposition 7.1. *Let \wp be the Weierstrass \wp -function with algebraic invariants g_2 and g_3 and with complex multiplication by the field k . If ω is any period of $\wp(z)$, η is the corresponding quasi-period and τ is any element of k which is not real, then each of the sets*

$$\{\pi, \omega, e^{2\pi i\tau}\}, \quad \{\pi, \eta, e^{2\pi i\tau}\}$$

is algebraically independent.

For a Weierstrass \wp -function with algebraic invariants g_2 and g_3 and field of endomorphisms k , the set

$$\mathcal{L}_E = \{\alpha \in \mathbb{C} : \wp(\alpha) \in \overline{\mathbb{Q}} \cup \{\infty\}\}$$

is referred to as the set of elliptic logarithms of algebraic points on E . Here E is the associated elliptic curve. Let Ω be the lattice of periods. This k -linear space \mathcal{L}_E is the elliptic analog of the \mathbb{Q} -linear space of logarithms of non-zero algebraic numbers for the exponential case. The question of linear independence of elliptic logarithms, analogous to Baker's theorem, has been established by Masser for the CM case [13] and by Bertrand and Masser for the non-CM case [5].

The algebraic independence of the values of the Weierstrass \wp -function is more delicate. When the Weierstrass \wp -function has complex multiplication, the following analogue of the Lindemann-Weierstrass Theorem has been proved by Philippon [16] and Wüstholz [22].

Theorem 7.2 (Philippon and Wüstholz). *Let $\wp(z)$ be a Weierstrass \wp -function with algebraic invariants g_2 and g_3 that has complex multiplication. Let k be its field of endomorphisms. Let*

$$\alpha_1, \alpha_2, \dots, \alpha_n$$

be algebraic numbers which are linearly independent over k . Then the numbers $\wp(\alpha_1), \dots, \wp(\alpha_n)$ are algebraically independent.

For the non-CM case, so far only the algebraic independence of at least $n/2$ of these numbers is known by the work of Chudnovski [6].

Motivated by the results of Nesterenko and of Philippon and Wüstholz, we ask the following question, which can be regarded as an elliptic-exponential extension of the conjecture of Schanuel.

Question. Let $\wp(z)$ be a Weierstrass \wp -function with algebraic invariants g_2 and g_3 and lattice Ω . Let k be its field of endomorphisms. Let

$$\alpha_1, \alpha_2, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_n$$

be complex numbers which are linearly independent over k and are not in Ω . Then is it true that the transcendence degree of the field

$$\mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_r}, \wp(\alpha_{r+1}), \dots, \wp(\alpha_n))$$

over \mathbb{Q} is at least n ?

This conjecture is a special case of a more general conjecture of Bertolin [4]. Indeed, we can specialize "conjecture elliptico-torique" on p. 206 of [4] to the case of a single elliptic curve. It is then not difficult to see that Bertolin's conjecture reduces to the assertion that with $\alpha_1, \dots, \alpha_n$ as above, the transcendence degree of

$$\mathbb{Q}(\alpha_1, \alpha_2, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_r}, \wp(\alpha_{r+1}), \dots, \wp(\alpha_n), Z(\alpha_{r+1}), \dots, Z(\alpha_n))$$

over \mathbb{Q} is at least $r + 2(n - r) = 2n - r$, where here $Z(s)$ denotes the associated Weierstrass zeta function. (To see this, one needs to observe that the d_{i1} in [4] give rise to $Z(\alpha_i)$ coming from the elliptic integrals of the second kind.) In particular, our question is a special case of Bertolin's conjecture.

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