ACTA ARITHMETICA
155.3 (2012)

Linear independence of Hurwitz zeta values and a theorem of Baker–Birch–Wirsing over number fields

by

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To Professor Schinzel on his 75th birthday

1. Introduction. For $x \in \mathbb{R}$ with $0 < x \leq 1$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, one defines the Hurwitz zeta function as

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.$$

For $x = 1$, note that $\zeta(s, 1)$ is the classical Riemann zeta function. Let $q, k > 1$ be integers and consider the $\varphi(q)$ numbers

$$\zeta(k, a/q), \quad (a, q) = 1, \, 1 \leq a \leq q.$$

The $\mathbb{Q}$-linear independence of these numbers, suggested by Chowla and Milnor, is linked to irrationality of zeta values and has been investigated in an earlier work [5]. In this work, we attempt to extend our investigation to linear independence over number fields.

Let $\mathbb{F}$ be a number field. Let us define the following $\mathbb{F}$-linear spaces:

DEFINITION 1. Let $q > 1$ be an integer. For integers $k > 1$, let $\mathrm{V}_k(q, \mathbb{F})$ be the $\mathbb{F}$-linear space defined by

$$\mathrm{V}_k(q, \mathbb{F}) = \mathbb{F}\text{-span of } \{\zeta(k, a/q) : 1 \leq a < q, \, (a, q) = 1\}.$$

We want to study the dimension of this space. At the outset, we note that this dimension, for fixed q and k, depends on the number field $\mathbb{F}$. In other words, the dimension can be different for different choices of the base field $\mathbb{F}$.

Suppose that $\mathbb{F}$ is the qth cyclotomic field $\mathbb{Q}(\zeta_q)$. Then we have the following upper bound.

2010 *Mathematics Subject Classification*: 11J86, 11M35, 11R18, 11R21, 11R32.
Key words and phrases: Hurwitz zeta values, polylogarithms, non-vanishing of $L(s, f)$.

DOI: 10.4064/aa155-3-6

Proposition 1.1. *The dimension of the space* $\mathrm{V}_k(q, \mathbb{Q}(\zeta_q))$ *is at most* $\varphi(q)/2 + 1$.

Though we do not have a non-trivial lower bound for the above dimension, we have the following conditional lower bound.

Proposition 1.2. *There exists an integer* $r > 1$ *such that for all integers* $q > 2$ *which are co-prime to* r *and all odd integers* $k > 1$, *the dimension of the space* $\mathrm{V}_k(q, \mathbb{Q}(\zeta_q))$ *is at least* 2.

However, for integers $k, q > 1$ and number fields $\mathbb{F}$ such that $\mathbb{Q}(\zeta_q) \cap \mathbb{F} = \mathbb{Q}$, we expect a different answer. More precisely, we expect that the dimension of the space $\mathrm{V}_k(q, \mathbb{F})$ in this case is equal to $\varphi(q)$. Here, we have the following lower bound:

Theorem 1.3. *Let* $q > 1$ *be an integer and* $\mathbb{F}$ *be a number field such that* $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. *Then*

$$\dim_{\mathbb{F}} \mathrm{V}_k(q, \mathbb{F}) \geq \varphi(q)/2 \quad \textit{for integers } k > 1.$$

Any improvement of the above lower bound would have non-trivial consequences. For instance, we prove the following theorems in Section 5.

Theorem 1.4. *Let* $\mathbb{F}$ *be a number field such that* $\mathbb{F} \cap \mathbb{Q}(i) = \mathbb{Q}$ *and* $k > 1$ *be an odd integer. Then* $\dim_{\mathbb{F}} \mathrm{V}_k(4, \mathbb{F}) = 2$ *for all such* $\mathbb{F}$ *implies that* $\zeta(k)/\pi^k$ *is transcendental.*

Theorem 1.5. *Let* $\mathbb{F}$ *be a number field such that* $\mathbb{F} \cap \mathbb{Q}(\zeta_3) = \mathbb{Q}$ *and* $k > 1$ *be an odd integer. Then*

$$\dim_{\mathbb{F}} \mathrm{V}_k(3, \mathbb{F}) = 2 \quad \textit{is equivalent to} \quad \zeta(k)/\sqrt{3}\,\pi^k \notin \mathbb{F}.$$

In this connection, we prove the following theorem.

Theorem 1.6. *Let* $k > 1$ *be an odd integer and* $q, r > 2$ *be two co-prime integers. Also, let* $\mathbb{F}$ *be a subfield of the real numbers such that* $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q} = \mathbb{F} \cap \mathbb{Q}(\zeta_r)$ *and also* $\mathbb{F}(\zeta_q) \cap \mathbb{F}(\zeta_r) = \mathbb{F}$. *Then either*

$$\dim_{\mathbb{F}} \mathrm{V}_k(q, \mathbb{F}) \geq \varphi(q)/2 + 1 \quad \textit{or} \quad \dim_{\mathbb{F}} \mathrm{V}_k(r, \mathbb{F}) \geq \varphi(r)/2 + 1.$$

As an immediate corollary, we have for $\mathbb{F}$ as above:

Corollary 1.7. *Let* k *be an odd integer. Then either* $\dim_{\mathbb{F}} \mathrm{V}_k(3, \mathbb{F}) = 2$ *or* $\dim_{\mathbb{F}} \mathrm{V}_k(4, \mathbb{F}) = 2$.

The following theorem suggests a recipe for constructing number fields $\mathbb{F}$ of the type alluded to in Theorem 1.6.

Theorem 1.8. *Let* $\mathbb{F}$ *be a finite Galois extension of* $\mathbb{Q}$ *with discriminant* $d_{\mathbb{F}}$. *Also, let* $(d_{\mathbb{F}}, qr) = 1$, *where* $q, r > 1$ *with* $(q, r) = 1$. *Then* $\mathbb{F}(\zeta_q) \cap \mathbb{F}(\zeta_r) = \mathbb{F}$.

As we mentioned before, we believe a much stronger statement than Theorem 1.6 should be true. More precisely:

Conjecture 1. Let $q > 1$ be an integer and $\mathbb{F}$ be a number field such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then $\dim_{\mathbb{F}} \mathrm{V}_k(q, \mathbb{F}) = \varphi(q)$ for all integers $k > 1$.

This conjecture can be thought of as a generalization of a conjecture of P. Chowla and S. Chowla [3] for $k = 2$ and its further generalization by Milnor [8] for all $k > 1$ (see [5] for further details).

We note that the linear independence of the Hurwitz zeta values $\zeta(k, a/q)$ for $k > 1$ is related to the non-vanishing of the L-series

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \qquad \Re(s) > 1,$$

at $s = k$, where f is a periodic function with period q. This link is established through the following identity:

$$L(s, f) = \frac{1}{q^s} \sum_{a=1}^{q} f(a)\zeta(s, a/q), \qquad \Re(s) > 1.$$

The question of non-vanishing of $L(1, f)$ when f is rational-valued was raised by Chowla. The work of Baker, Birch and Wirsing [1] gave a satisfactory answer to Chowla's question. In conformity with the generalization envisaged here for $k > 1$, we extend their investigation to more general number fields. More precisely, we derive the following generalization of the Baker–Birch–Wirsing Theorem in the penultimate section.

Theorem 1.9. *For an integer $q > 1$, let f be a periodic function with period q taking values in a number field $\mathbb{F}$. Further, $f(a) = 0$ whenever $1 < (a, q) < q$. Also, let $\mathrm{K} = \mathbb{F} \cap \mathbb{Q}(\zeta_q)$ and $\mathrm{H} = \mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathrm{K}) \subseteq (\mathbb{Z}/q\mathbb{Z})^*$. Assume that $\mathrm{supp}(f)$, the support of f in $\mathbb{Z}/q\mathbb{Z}$, is contained in $\mathrm{H} \cup \{q\}$. Then $L(1, f) = 0$ if and only if $f \equiv 0$.*

If $\mathrm{K} = \mathbb{Q}$, this is the original Baker–Birch–Wirsing theorem. We also apply this to derive linear independence of certain L-values associated to Dirichlet characters.

In the final section, we link the linear independence of the Hurwitz zeta values $\zeta(k, a/q)$ to the Polylog Conjecture formulated in [5]. Let us recall the definition of polylogs.

Definition 2. For an integer $k \geq 2$ and complex numbers $z \in \mathbb{C}$ with $|z| \leq 1$, the polylogarithm function $\mathrm{Li}_k(z)$ is defined by

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

For $k=1$, the series is $-\log(1-z)$ provided $|z|\le 1, z\ne 1$. Analogous to Baker's theorem on linear forms in logarithms, the following conjecture about polylogarithms was formulated in [5].

Polylog Conjecture. Suppose that $\alpha_1,\dots,\alpha_n$ are algebraic numbers with absolute values $|\alpha_i|\le 1$ such that $\mathrm{Li}_k(\alpha_1),\dots,\mathrm{Li}_k(\alpha_n)$ are linearly independent over $\mathbb{Q}$. Then they are linearly independent over the field of algebraic numbers $\overline{\mathbb{Q}}$.

Apart from the case $k=1$, which is a special case of Baker's theorem, almost nothing is known about the above conjecture. We deduce the following theorem:

Theorem 1.10. *Assume that the Polylog Conjecture is true. Then Conjecture* 1 *is true.*

2. The case $\mathbb{F}=\mathbb{Q}(\zeta_q)$

Proof of Proposition 1.1. We have the following identity (see [10], for instance):

$$\zeta(k,a/q)+(-1)^k\zeta(k,1-a/q)=\frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}(\pi\cot\pi z)\bigg|_{z=a/q}. \tag{1}$$

Note that

$$\frac{d^{k-1}}{dz^{k-1}}(\pi\cot\pi z)=\pi^k\sum_{\substack{r,s\ge 0\\ r+2s=k}}\beta_{r,s}\cot^r\pi z\ (1+\cot^2\pi z)^s,$$

where $\beta_{r,s}\in\mathbb{Q}$. Since $i\cot\frac{\pi a}{q}\in\mathbb{Q}(\zeta_q)$, we see that

$$\zeta(k,a/q)+(-1)^k\zeta(k,1-a/q)=i^k\pi^k\alpha_{a,q}$$

where $\alpha_{a,q}\in\mathbb{Q}(\zeta_q)$. This proves Proposition 1.1.

Proof of Proposition 1.2. For co-prime integers q_j, $j=1,2$, and k odd, by the above observations we see that

$$\zeta(k,a/q_j)-\zeta(k,1-a/q_j)=i\pi^k\alpha_{a,q_j}$$

where $\alpha_{a,q_j}\in\mathbb{Q}(\zeta_{q_j})$. If the dimension of both the spaces $\mathrm{V}_k(q_j,\mathbb{Q}(\zeta_{q_j}))$ is 1, then

$$\frac{\zeta(k)}{i\pi^k}\in\mathbb{Q}(\zeta_{q_1})\cap\mathbb{Q}(\zeta_{q_2})=\mathbb{Q},\qquad \text{since } (q_1,q_2)=1.$$

This is a contradiction since $\zeta(k)/i\pi^k$ is a purely imaginary complex number.

3. The case $\mathbb{F}\cap\mathbb{Q}(\zeta_q)=\mathbb{Q}$ and proofs of Theorems 1.3 and 1.6.

We need the following theorem of Okada [11] (see also [4] and [10]).

LEMMA 3.1. *Let k and q be positive integers with $k > 0$ and $q > 2$. Let* T *be a set of $\varphi(q)/2$ representatives mod q such that the union* $\mathrm{T} \cup (-\mathrm{T})$ *constitutes a complete set of co-prime residue classes mod q. Let $\mathbb{F}$ be a number field such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then the set of real numbers*

$$\frac{d^{k-1}}{dz^{k-1}} \cot \pi z \bigg|_{z=a/q}, \qquad a \in T,$$

is linearly independent over $\mathbb{F}$.

The polylogarithm function $\mathrm{Li}_k(z)$ defined in the introduction for integers $k \geq 2$ and complex z with $|z| \leq 1$ can be extended for all integers k and for all z in $\mathbb{C} - [1, \infty)$. We refer to the paper of Milnor ([8, p. 285]) for details. Let, for real $x \neq 0$, $\mathrm{li}_k(x) := \mathrm{Li}_k(e^{2\pi i x})$. Then, since (see [8])

$$\frac{d}{dx} \mathrm{li}_k(x) = 2\pi i \, \mathrm{li}_{k-1}(x) \quad \text{and} \quad \mathrm{li}_0(x) = \frac{-1 + i \cot \pi x}{2},$$

Lemma 3.1 is an instance of linear independence of polylog values (for negative values of k) over certain number fields.

Proof of Theorem 1.3. Note that the space $\mathrm{V}_k(q, \mathbb{F})$ is spanned by the following set of real numbers:

$$\{\zeta(k, a/q) \pm \zeta(k, 1 - a/q) : (a, q) = 1, \, 1 \leq a < q/2\}.$$

Identity (1) along with Okada's result completes the proof.

Proof of Theorem 1.6. For the proof, we need a more refined analysis of the terms appearing on the right hand side of (1). A periodic function with period q is called *odd* if $f(a) = -f(q - a)$ for $1 \leq a \leq q$. Given any a with $(a, q) = 1$, let δ_a be the odd q-periodic function which takes the value 1 at a and is supported in $\{a, q - a\}$. Then

$$L(k, \delta_a) = \frac{1}{q^k} \sum_{b=1}^{q} \delta_a(b) \zeta(k, b/q) = \frac{1}{q^k} [\zeta(k, a/q) - \zeta(k, 1 - a/q)].$$

On the other hand,

$$2L(k, \delta_a) = \frac{(2\pi i)^k}{k!} \sum_{b=1}^{q} \widehat{\delta}_a(b) \mathrm{B}_k(b/q),$$

where

$$\mathrm{B}_k(x) = \frac{-k!}{(2\pi i)^k} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n x}}{n^k}$$

is the kth periodic Bernoulli polynomial and

$$\widehat{\delta}_a(n) = \frac{1}{q} \sum_{b=1}^{q} \delta_a(b) e^{2i\pi b n/q} = \frac{1}{q} [\zeta_q^{an} - \zeta_q^{-an}] \quad \text{with } \zeta_q = e^{2\pi i/q}.$$

Thus, we have

$$\frac{\zeta(k,a/q)-\zeta(k,1-a/q)}{(2\pi i)^k}=\frac{q^{k-1}}{2k!}\sum_{b=1}^{q}(\zeta_q^{ab}-\zeta_q^{-ab})\mathrm{B}_k(b/q)\in\mathbb{Q}(\zeta_q)\subseteq\mathbb{F}(\zeta_q).$$

Now, let q and r be two co-prime integers. Suppose that

$$\dim_{\mathbb{F}}\mathrm{V}_k(q,\mathbb{F})=\varphi(q)/2.$$

Since k is odd, the numbers

$$\zeta(k,a/q)-\zeta(k,1-a/q),\qquad\text{where } (a,q)=1,\ 1\le a<q/2,$$

generate $\mathrm{V}_k(q,\mathbb{F})$. Hence

$$\frac{\zeta(k)}{i\pi^k}=\sum_{\substack{(a,q)=1\\1\le a<q/2}}\lambda_a\frac{(\zeta(k,a/q)-\zeta(k,1-a/q))}{(2i\pi)^k}\in\mathbb{F}(\zeta_q),\qquad\text{where } \lambda_a\in\mathbb{F}.$$

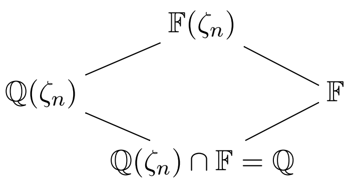
Similarly, if $\dim_{\mathbb{F}}\mathrm{V}_k(r,\mathbb{F})=\varphi(r)/2$, then $\zeta(k)/i\pi^k\in\mathbb{F}(\zeta_r)$. Hence

$$\frac{\zeta(k)}{i\pi^k}\in\mathbb{F}(\zeta_q)\cap\mathbb{F}(\zeta_r)=\mathbb{F}$$

as q and r are co-prime. This is a contradiction as $\mathbb{F}\subset\mathbb{R}$. Thus

$$\dim_{\mathbb{F}}\mathrm{V}_k(q,\mathbb{F})\ge\varphi(q)/2+1\quad\text{or}\quad\dim_{\mathbb{F}}\mathrm{V}_k(r,\mathbb{F})\ge\varphi(r)/2+1.$$

4. Proof of Theorem 1.8. Let $\mathbb{F}$ be a finite Galois extension of $\mathbb{Q}$ with discriminant $d_{\mathbb{F}}$. Suppose that for integers $q,r>1$, $(d_{\mathbb{F}},qr)=1$. Hence $\mathbb{F}\cap\mathbb{Q}(\zeta_q)=\mathbb{Q}$ and $\mathbb{F}\cap\mathbb{Q}(\zeta_r)=\mathbb{Q}$. Consider the diagram



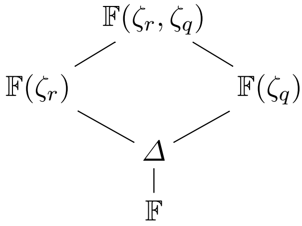
where $n=q$ or r. Since $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is Galois, also $\mathbb{F}(\zeta_n)/\mathbb{F}$ is Galois and (for details, see [6, p. 266])

$$\mathrm{Gal}(\mathbb{F}(\zeta_n)/\mathbb{F})\simeq\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}).$$

Hence $[\mathbb{F}(\zeta_n):\mathbb{F}]=\varphi(n)$ for $n=q,r$. Now set $\Delta=\mathbb{F}(\zeta_r)\cap\mathbb{F}(\zeta_q)$. We want to show that $\Delta=\mathbb{F}$. To do this, we just need to compare degrees. Clearly,

$$[\mathbb{F}(\zeta_r):\mathbb{F}]\ge[\mathbb{F}(\zeta_r):\Delta].$$

But we have



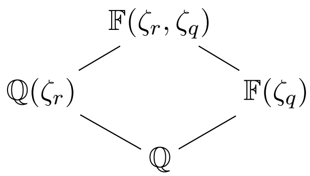
All these extensions are Galois extensions and we have

$$\mathrm{Gal}(\mathbb{F}(\zeta_r)/\Delta) \simeq \mathrm{Gal}(\mathbb{F}(\zeta_r, \zeta_q)/\mathbb{F}(\zeta_q))$$

so that

$$[\mathbb{F}(\zeta_r) : \Delta] = [\mathbb{F}(\zeta_r, \zeta_q) : \mathbb{F}(\zeta_q)]. \tag{2}$$

Since $(d_{\mathbb{F}}, qr) = 1 = (q, r)$ and every non-trivial extension of $\mathbb{Q}$ is ramified, we have $\mathbb{Q}(\zeta_r) \cap \mathbb{F}(\zeta_q) = \mathbb{Q}$ by ramification considerations. Now, from the diagram



it follows that $[\mathbb{F}(\zeta_r, \zeta_q) : \mathbb{F}(\zeta_q)] = [\mathbb{Q}(\zeta_r) : \mathbb{Q}] = \varphi(r)$. Thus, returning to (2) with this information we have

$$[\mathbb{F}(\zeta_r) : \Delta] = [\mathbb{F}(\zeta_r, \zeta_q) : \mathbb{F}(\zeta_q)] = \varphi(r).$$

But $[\mathbb{F}(\zeta_r) : \mathbb{F}] = \varphi(r)$. Hence $\Delta = \mathbb{F}$. This completes the proof of the theorem.

5. Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4. Let $\mathbb{F}$ be a number field such that $\mathbb{F} \cap \mathbb{Q}(i) = \mathbb{Q}$. Since $k > 1$ is an odd integer, by equation (1), we have

$$\zeta(k, 1/4) - \zeta(k, 3/4) = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z)|_{z=1/4},$$

where $\frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z)\big|_{z=1/4}$ is a rational multiple of π^k. Also,

$$\zeta(k, 1/4) + \zeta(k, 3/4) = (4^k - 2^k)\zeta(k).$$

Hence by Lemma 3.1, $\dim_{\mathbb{F}} \mathrm{V}_k(4, \mathbb{F}) = 2$ is equivalent to $\zeta(k)/\pi^k \notin \mathbb{F}$.

Proof of Theorem 1.5. Let $\mathbb{F}$ be a number field such that $\mathbb{F}\cap\mathbb{Q}(\zeta_3)=\mathbb{Q}$. Since $k>1$ is an odd integer, by equation (1), we have

$$\zeta(k,1/3)-\zeta(k,2/3)=\frac{1}{(k-1)!}\,\frac{d^{k-1}}{dz^{k-1}}(\pi\cot\pi z)\Big|_{z=1/3},$$

where $\frac{d^{k-1}}{dz^{k-1}}(\pi\cot\pi z)\big|_{z=1/3}$ is a rational multiple of $\sqrt{3}$ and π^k. Also,

$$\zeta(k,1/3)+\zeta(k,2/3)=(3^k-1^k)\zeta(k).$$

Hence by Lemma 3.1, $\dim_{\mathbb{F}}\mathrm{V}_k(3,\mathbb{F})=2$ is equivalent to $\zeta(k)/\sqrt{3}\,\pi^k\notin\mathbb{F}$.

6. Proof of Theorem 1.9 and applications. For the proof of Theorem 1.9, we shall need the following lemma (see [6, p. 548]).

Lemma 6.1. *Let G be a finite abelian group of order n and $F:G\to\mathbb{C}$ be a complex-valued function on G. Also, let* B *be the Dedekind matrix $(F(xy^{-1}))_{n\times n}$. Then*

$$\det(\mathrm{B})=\prod_{\chi}\Big(\sum_{x\in G}\chi(x)F(x)\Big),$$

where the product is over all characters χ of G.

Proof of Theorem 1.9. Recall that the digamma function $\psi(z)$ for $z\neq -n$, where $n\in\mathbb{N}$, is the logarithmic derivative of the Γ-function and is given by

$$-\psi(z)=\gamma+\frac{1}{z}+\sum_{n\geq 1}\left(\frac{1}{n+z}-\frac{1}{n}\right).$$

As shown in [9], if $\sum_{a\in\mathrm{H}\cup\{q\}}f(a)=0$ then $L(1,f)$ exists and

$$L(1,f)=\sum_{n=1}^{\infty}\frac{f(n)}{n}=\frac{-1}{q}\sum_{a\in\mathrm{H}}f(a)(\psi(a/q)+\gamma).$$

Here we have used the fact that $f(q)=-\sum_{a\in\mathrm{H}}f(a)$ and that $\psi(1)=-\gamma$. Also

$$L(1,f)=-\sum_{a=1}^{q-1}\widehat{f}(a)\log(1-\zeta_q^a),$$

where

$$\widehat{f}(a)=\frac{1}{q}\sum_{n=1}^{q}f(n)\zeta_q^{-an}$$

is the Fourier transform of f. Let

$$\log(1-\zeta_q^{\alpha_1}),\ldots,\log(1-\zeta_q^{\alpha_t})$$

be a maximal $\mathbb{F}$-linearly independent subset of

$$\{\log(1-\zeta_q^a):1\leq a\leq q-1\}.$$

Write

$$\log(1-\zeta_q^a) = \sum_{b=1}^{t} \mathrm{A}_{ab} \log(1-\zeta_q^{\alpha_b}),$$

where $\mathrm{A}_{ab} \in \mathbb{F}$. Then by the given hypothesis $L(1,f)=0$, we have

$$\beta_1 \log(1-\zeta_q^{\alpha_1}) + \cdots + \beta_t \log(1-\zeta_q^{\alpha_t}) = 0$$

where

$$\beta_b = \sum_{a=1}^{q-1} \widehat{f}(a)\mathrm{A}_{ab}.$$

Since f takes values in $\mathbb{F}$, $\widehat{f}$ is algebraic-valued. Thus by Baker's theorem on linear forms in logarithms, we have

$$\beta_b = \sum_{a=1}^{q-1} \widehat{f}(a)\mathrm{A}_{ab} = 0, \qquad 1 \le b \le t.$$

Then for any automorphism $\sigma \in \mathrm{Gal}(\mathbb{F}(\zeta_q)/\mathbb{F})$, we have

$$\sum_{a=1}^{q-1} \sigma(\widehat{f}(a))\mathrm{A}_{ab} = 0, \qquad 1 \le b \le t,$$

and hence

$$\sum_{a=1}^{q-1} \sigma(\widehat{f}(a)) \log(1-\zeta_q^a) = 0.$$

Since $\mathrm{Gal}(\mathbb{F}(\zeta_q)/\mathbb{F}) \simeq \mathrm{H}$ by means of the restriction map, for $h \in \mathrm{H}$, let $\sigma_h \in \mathrm{Gal}(\mathbb{F}(\zeta_q)/\mathbb{F})$ be such that

$$\sigma_h(\zeta_q) = \zeta_q^h.$$

Define $f_h(n) := f(nh^{-1})$ for $h \in \mathrm{H}$. Then we have

$$\sum_{a\in \mathrm{H}} f_h(a) = -f_h(q) = -f(q) \quad \text{and} \quad \sigma_h(\widehat{f}(n)) = \widehat{f_h}(n).$$

Hence

$$\begin{aligned} L(1,f_h) = \sum_{n=1}^{\infty} \frac{f_h(n)}{n} &= -\sum_{a=1}^{q-1} \widehat{f_h}(a) \log(1-\zeta_q^a) \\ &= -\sum_{a=1}^{q-1} \sigma_h(\widehat{f}(a)) \log(1-\zeta_q^a) = 0 \end{aligned}$$

for all $h \in \mathrm{H}$. This gives

$$\frac{-1}{q} \sum_{a\in \mathrm{H}} f_h(a)(\psi(a/q)+\gamma) = L(1,f_h) = 0.$$

Hence by making a change of variable, we have

$$\sum_{a\in \mathrm{H}} f(a)(\psi(\overline{ah}/q)+\gamma)=0, \tag{3}$$

where we use the notation $\overline{ah}$ to indicate the reduced residue class $b \pmod q$ satisfying

$$ah \equiv b \pmod q.$$

Now

$$\mathrm{A} := (\psi(\overline{ah}/q)+\gamma)_{a,h\in \mathrm{H}}$$

is a Dedekind matrix and its determinant is given by

$$\prod_{\chi\in\widehat{\mathrm{H}}}\Big(\sum_{h\in\mathrm{H}}\chi(h)(\psi(h/q)+\gamma)\Big).$$

By Pontryagin duality, there is a unique subgroup $\mathrm{V} \subseteq (\mathbb{Z}/q\mathbb{Z})^*$ such that

$$\widehat{\mathrm{H}} \simeq (\mathbb{Z}/q\mathbb{Z})^*/\mathrm{V}.$$

See [12, Chapter 3] for details. Thus, there is a unique extension $\mathrm{M_V}/\mathbb{Q}$ such that

$$\mathbb{Q} \subseteq \mathrm{M_V} \subseteq \mathbb{Q}(\zeta_q) \quad \text{and} \quad \mathrm{Gal}(\mathrm{M_V}/\mathbb{Q}) \simeq \mathrm{H}.$$

Now the characters χ of $\mathrm{Gal}(\mathrm{M_V}/\mathbb{Q})$ extend to $\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$ and so we may identify, for non-principal characters χ,

$$\sum_{h\in\mathrm{H}}\chi(h)(\psi(h/q)+\gamma) = -qL(1,\chi)$$

since the extended characters are Dirichlet characters in the classical sense. (This is a special case of a general property of Artin L-series; see for example, property L3 on p. 233 in [7].) Since $L(1,\chi) \neq 0$ for $\chi \neq 1$, we need only verify that

$$\sum_{h\in\mathrm{H}}(\psi(h/q)+\gamma) \neq 0.$$

Since $\psi(x)$ is an increasing function and $\psi(1) = -\gamma$, we have the above identity. This completes the proof of Theorem 1.9.

COROLLARY 6.2. *Let* $\mathrm{K} = \mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_{\phi(q)})$. *Put* $\mathrm{N} = \mathrm{Gal}(\mathrm{K}/\mathbb{Q})$. *Then every character of* N *extends to a Dirichlet character of* $(\mathbb{Z}/q\mathbb{Z})^* \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$. *Let* $\chi_1,\ldots,\chi_r$ *be the Dirichlet characters of* $\widehat{\mathrm{N}}$. *Choose representatives* $X_1,\ldots,X_t$ *of* $\widehat{(\mathbb{Z}/q\mathbb{Z})^*}/\widehat{\mathrm{N}}$. *Let* $\mathrm{V}_i = \sum_{j=1}^r X_i\chi_j$ *and consider the values*

$$L(1,\mathrm{V}_i) = \sum_{j=1}^{r} L(1, X_i\chi_j), \qquad 1 \le i \le t,\ X_i\chi_j \neq 1.$$

Then these values are linearly independent over K.

REMARK. In the case $\mathrm{K} = \mathbb{Q}$, we retrieve the theorem of Baker–Birch–Wirsing that $L(1, \chi)$ as χ varies over all non-trivial Dirichlet characters are linearly independent over $\mathbb{Q}$ when $(q, \phi(q)) = 1$.

Proof of Corollary 6.2. Assume that

$$\sum_{i=1}^{t} c_i L(1, \mathrm{V}_i) = 0 \tag{4}$$

with $c_i \in \mathrm{K}$. Put

$$f = \sum_{i=1}^{t} c_i X_i \sum_{j=1}^{r} \chi_j.$$

Then equation (4) says that $L(1, f) = 0$. Since X_i, χ_j are Dirichlet characters mod q, they take values in $\mathbb{Q}(\zeta_{\phi(q)})$. We first show that

$$\mathrm{supp}(f) \subseteq \mathrm{H} \tag{5}$$

where $\mathrm{H} = \mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathrm{K})$. Observe that $\mathrm{N} \simeq (\mathbb{Z}/q\mathbb{Z})^*/\mathrm{H}$. In other words, characters of N are Dirichlet characters mod q which are trivial on H. We need to show that $f(a) = 0$ if $a \notin \mathrm{H}$. By the orthogonality relations,

$$\chi_1(a) + \cdots + \chi_r(a) = \begin{cases} |\mathrm{N}|, & a \in \mathrm{H}, \\ 0, & a \notin \mathrm{H}, \end{cases}$$

for the group $\mathrm{N} \simeq (\mathbb{Z}/q\mathbb{Z})^*/\mathrm{H}$. Thus,

$$f(a) = \sum_{i=1}^{t} c_i X_i(a) \sum_{j=1}^{r} \chi_j(a)$$

and the inner sum is zero unless $a \in \mathrm{H}$. This proves assertion (5).

So we can apply our theorem to deduce $f \equiv 0$. In other words,

$$\sum_{j=1}^{r} \sum_{i=1}^{t} c_i X_i \chi_j \equiv 0.$$

One needs to observe that $X_i\chi_j$, $1 \le j \le r$, $1 \le i \le t$, are distinct Dirichlet characters mod q. By the linear independence of characters, we see that $c_i = 0$ for all i. This proves the corollary.

7. Proof of Theorem 1.10. We shall now give a proof of Theorem 1.10 following the methodology employed in [1].

Proof of Theorem 1.10. Let $k, q > 1$ be integers and $\mathbb{F}$ be a number field such that $\mathbb{F} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Note that Conjecture 1 is equivalent to the assertion that $L(k, f) \neq 0$, where $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{F}$ with $f(a) = 0$ whenever

$1 < (a,q) \leq q$ and $f \not\equiv 0$. This is again because of the identity

$$L(k,f) = \frac{1}{q^k} \sum_{a=1}^{q} f(a)\zeta(k, a/q).$$

For simplicity, we prove Theorem 1.10 when $q = p$ is a prime. The proof of the general case is identical.

For an $\mathbb{F}$-valued periodic function f with prime period p and $f(p) = 0$, suppose that $L(k,f) = 0$. Then

$$L(k,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^k} = \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{a=1}^{p} \widehat{f}(a)\zeta_p^{an} = \sum_{a=1}^{p} \widehat{f}(a) \operatorname{Li}_k(\zeta_p^a) = 0,$$

where $\widehat{f}$ is the Fourier inversion of f. Then choosing a maximal $\mathbb{F}$-linearly independent subset of the above polylogarithms and arguing exactly as in the proof of Theorem 1.9, we deduce that for any automorphism σ in the Galois group of $\mathbb{F}(\zeta_p)$ over $\mathbb{F}$,

$$\sum_{a=1}^{p} \sigma(\widehat{f}(a)) \operatorname{Li}_k(\zeta_p^a) = 0.$$

For $1 \leq h \leq p-1$, let $\sigma_h \in \operatorname{Gal}(\mathbb{F}(\zeta_p)/\mathbb{F})$ be such that

$$\sigma_h(\zeta_p) = \zeta_p^h.$$

Define $f_h(n) := f(nh^{-1})$ for $1 \leq h \leq p-1$. Then we have

$$\sigma_h(\widehat{f}(n)) = \widehat{f_h}(n).$$

Hence

$$L(k, f_h) = \sum_{n=1}^{\infty} \frac{f_h(n)}{n^k} = p^{-k} \sum_{a=1}^{p-1} f_h(a)\zeta(k, a/p) = 0.$$

A change of variable gives

$$L(k, f_h) = p^{-k} \sum_{a=1}^{p-1} f(a)\zeta(k, ah/p) = 0 \tag{6}$$

for all $1 \leq h \leq p-1$. We treat this as a matrix equation with coefficient matrix B being the $(p-1) \times (p-1)$ matrix whose (a,h)th entry is given by $\zeta(k, ah/p)$. Then we have, by Lemma 6.1,

$$\operatorname{Det}(\mathrm{B}) = \pm \prod_{\chi} p^k L(k,\chi) \neq 0.$$

Thus the matrix B is invertible and hence by equation (6) we have $f \equiv 0$. This completes the proof in the case when $q = p$ is a prime. The proof for an arbitrary modulus q is identical, the final determinant being associated to the group $(\mathbb{Z}/q\mathbb{Z})^{\times}$.

Acknowledgments. The authors would like to thank the Fields Institute for providing a congenial environment where this work got started. We are also grateful to the referee for his/her very careful reading and several helpful suggestions.

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Received on 14.9.2011
and in revised form on 3.6.2012 (6826)