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On Zagier's cusp form and the Ramanujan τ function

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Dedicated to the memory of Professor K G Ramanathan

Abstract. Zagier constructed a cusp form for each weight k of the full modular group. We use this construction to estimate Fourier coefficients of cusp forms. In particular, we get a non-trivial estimate, by elementary methods and indicate a relationship with the Lindelof hypothesis for classical Dirichlet L-functions.

Keywords. Ramanujan tau function; cusp forms.

In [Z], Zagier constructs a non-zero cusp form of weight k for the full modular group for every even integer $k \geqslant 12$. More precisely, define for real numbers Δ and r satisfying $\Delta < r^2$ and $s \in \mathbb{C}$ with $1/2 < \mathrm{Re}(s) < k$,

$$I_k(\Delta, r; s) = \int_0^\infty \int_{-\infty}^\infty \frac{y^{k+s-2}\mathrm{d}x\,\mathrm{d}y}{(x^2 + y^2 + iry - \Delta/4)^k}.$$

It is easily seen that if $\Delta \neq 0$, we can simplify this to

$$I_k(\Delta, r; s) = \frac{\Gamma(k - 1/2)\Gamma(1/2)}{\Gamma(k)} \int_0^\infty \frac{y^{k+s-2}\mathrm{d}y}{(y^2 + iry - \Delta/4)^{k-1/2}}.$$

The advantage of the last expression is that the integral is absolutely convergent for $1 - k < \mathrm{Re}(s) < k$. Moreover, it can be expressed in terms of Legendre functions.

Now let Δ be any discriminant. That is $\Delta \in \mathbb{Z}$ and $\Delta \equiv 0$ or $1 \pmod 4$. Consider the binary quadratic forms

$$\phi(u, v) = au^2 + buv + cv^2 \quad (a, b, c \in \mathbb{Z})$$

with discriminant $|\phi| = b^2 - 4ac = \Delta$. The group $\Gamma = SL_2(\mathbb{Z})$ operates on the set of such forms by

$$(\gamma\phi)(u, v) = \phi(au + cv, bu + \mathrm{d}v)$$

when

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

When $\Delta \neq 0$, the number of Γ-equivalence classes is finite and we denote this number by $H(\Delta)$. Define for $\mathrm{Re}(s) > 1$.

$$\zeta(s,\Delta) = \sum_{\substack{\phi \bmod \Gamma \\ |\phi| = \Delta}} \sum_{\substack{(m,n)\in \mathbb{Z}^2/\Gamma_\phi \\ \phi(m,n) > 0}} \phi(m,n)^{-s},$$

where the first sum is over all Γ-equivalence classes of forms ϕ of discriminant Δ and the second is over inequivalent pairs of integers with respect to the stabilizer Γ_ϕ of ϕ. If Δ is the discriminant of a real or imaginary quadratic field K, then $\zeta(s,\Delta) = \zeta(s)$ $L(s,\Delta)$ where $\zeta(s)$ is the Riemann zeta function and $L(s,\Delta)$ is the classical Dirichlet series attached to the quadratic character $(\Delta/.)$. If $\Delta = Df^2$ where D equals 1 or the discriminant of a quadratic field and f is a natural number, then $\zeta(s,\Delta)$ differs from $\zeta(s,D)$ only by a finite Dirichlet series. Therefore, in all cases, we can write

$$\zeta(s,\Delta) = \zeta(s)\, L(s,\Delta)$$

and this defines $L(s,\Delta)$.

Define, for each natural number m, the function $F(m,s)$ as equal to zero if m is not a perfect square and if $m = u^2, u > 0$, then

$$F(m,s) = (-1)^{k/2} \frac{\Gamma(k+s-1)\zeta(2s)}{2^{2s+k-3}\pi^s \Gamma(k)} u^{k+s-1}.$$

Then Zagier [Z, p. 110] proved the following:

PROPOSITION

For $m = 1, 2, \ldots$ and $s \in \mathbb{C}$, set

$$c_m(s) = F(m,s) + m^{k-1} \sum_{r=-\infty}^{\infty} [I_k(r^2 - 4m, r; s) + I_k(r^2 - 4m, -r; s)] \\ L(s, r^2 - 4m).$$

Then,

(a) *the series defining $c_m(s)$ converges absolutely and uniformly for $2 - k < \mathrm{Re}(s) < k - 1$;*
(b) *the function*

$$\Phi_s(z) = \sum_{m=1}^{\infty} c_m(s) e^{2\pi i m z}$$

for $\mathrm{Im}(z) > 0$ and $2 - k < \mathrm{Re}(s) < k - 1$ is a cusp form of weight k for the full modular group;
(c) *if f is a normalized, cuspidal, Hecke eigenform, then*

$$(\Phi_s, f) = \frac{(-1)^{k/2}\pi\Gamma(s+k-1)}{2^{k-3}(k-1)(4\pi)^{s+k-1}} D_f(s+k-1)$$

where $(\cdot,\cdot)$ denotes the Petersson inner product and $D_f(s)$ is defined as follows. Let

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

be the Fourier expansion about the cusp $i\infty$. Then,

$$D_f(s) = \frac{\zeta(2s-2k+2)}{\zeta(s-k+1)} \sum_{n=1}^{\infty} \frac{a(n)^2}{n^s}.$$

Thus, Zagier's cusp form has many virtues. We see from (c) that Φ_s is not identically zero whenever the space of cusp forms is non-zero and so, the formula for $c_m(s)$ gives an explicit formula for the Fourier coefficient of a cusp form of weight k. (In particular, for $k = 12$, we get an explicit formula for $\tau(m)$.) Indeed, specializing at $s = 1$ gives the classical Eichler-Selberg trace formula. Moreover, (c) gives the analytic continuation of the "symmetric square L-function" attached to f. That is, if π_f is the associated automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$, then $D_f(s)$ is essentially the Langlands L-function $L(s, \mathrm{Sym}^2(\pi_f))$, which we know is an L-function attached to a cuspidal automorphic representation of $GL_3(\mathbb{A}_\mathbb{Q})$, by the work of Gelbart-Jacquet [GJ].

Our purpose here is to indicate that Zagier's cusp form can be used to derive estimates for the Ramanujan τ-function (and other coefficients of cusp forms of small weight). Of course, Deligne [D] proved that

$$\tau(n) = O(n^{11/2+\epsilon}).$$

Moreover, we know from [Mu] that this result is best possible. However, the work of Deligne [D] uses deep methods of algebraic geometry. In [L], Langlands suggests an analytic approach which requires analytic continuation of all the symmetric power L-functions, $L(s, \mathrm{Sym}^r(\pi_f))$ for a fixed half-plane. At present, such an analytic continution exists only for $r \leqslant 5$ by the work of Hecke [H], Shimura [S] and Shahidi [Sh]. Shahidi's theorems lead to

$$\tau(p) = O(p^{11/2+1/5})$$

for every prime number p. (It should be stressed that the method of Langlands addresses both holomorphic and non-holomorphic modular forms and not just the holomorphic ones to which the Deligne estimate applies). However, even these results require a great amount of representation theoretic background. Perhaps, the "simplest" method is that of Selberg [Se] using the method of Poincaré series for the explicit construction of cusp forms. His method leads to

$$\tau(n) = O(n^{11/2+1/4+\varepsilon}).$$

We believe that Zagier's cusp form has important uses in number theory (and the optimal one not being the use we are going to make below). We will use it to derive a quick estimate for $\tau(p)$. We will prove

Theorem. $\tau(p) = O(p^{11/2+7/16+\varepsilon})$.

Remark. The same estimate is valid for $\tau(n)$ as well. In fact, the method below is valid *mutatis mutandis* for n not a perfect square. When n is a perfect square, the $F(n,s)$ term is $O(n^{11/2+1/4+\varepsilon})$ for $s = 1/2 + \delta + it$ and $\delta > 0$ and arbitrarily small. The bound $\tau(p) = O(p^6)$ is straightforward and derived in any book on modular forms. We also remark that it should be possible to obtain the Selberg bound by the methods of this paper. Indeed, if we assume the Lindelöf hypothesis for $L(s, \Delta)$, it is easily seen that the Selberg estimate follows. Since this hypothesis is being applied only "on average" rather than to a single L-function, such a result should follow with a little more technical refinement.

We will need various estimates for the Dirichlet L-function in the critical strip.

Lemma 1. $L(1/2+it,\Delta) \ll (|\Delta|(|t|+2))^{3/16+\varepsilon}$

Proof. For fundamental discriminants and $t=0$ this is a result of Burgess [B] and the hybrid estimate is due to Heath-Brown [HB]. So we need only prove this for non-fundamental discriminants. Writing $\Delta = Df^2$, where D is a fundamental discriminant, we see that ([Z, p. 130])

$$L(s,\Delta) = L(s,D)\sum_{d/f}\mu(d)\left(\frac{D}{d}\right)d^{-s}\sigma_{1-2s}(f/d)$$

where

$$\sigma_w(n) = \sum_{d/n} d^w$$

When $s = 1/2+it$, we find

$$L(1/2+it,\Delta) \ll (|\Delta|(|t|+2))^{3/16+\varepsilon}$$

which gives the desired result.

By the Phragmén-Lindelöf principle (see [Ra]) we deduce that:

Lemma 2. *For* $0 \leqslant \delta < 1/2$,

$$L(1/2+\delta+it,\Delta) \ll (|\Delta|(|t|+2))^{(3/16+\varepsilon)(1-2\delta)}.$$

We can now prove the theorem. Choose $s = 1/2+\delta$ with $\delta > 0$ and sufficiently small. Then

$$|c_p(1/2+\delta)| \leqslant p^{k-1}\sum_{r=-\infty}^{\infty} |I_k(\Delta,r,1/2+\delta)+I_k(\Delta,-r,1/2+\delta)|$$
$$|L(1/2+\delta,\Delta)|.$$

The integrals can be estimated without difficulty. Indeed,

$$|I_k(\Delta,r,1/2+\delta)| \leqslant \int_0^\infty \frac{y^{k-3/2+\delta}\mathrm{d}y}{(y^4+(r^2/2+2p)y^2+\Delta^2/16)^{k/2-1/4}}.$$

This last integral is bounded by

$$\int_0^\infty \frac{y^{k-3/2+\delta}\mathrm{d}y}{(y^4+(r^2/2+2p)y^2)^{k/2-1/4}},$$

which is readily evaluated upon substituting $y = (r^2/2+2p)^{1/2}u$, as

$$= (r^2/2+2p)^{-k/2+1/4+\delta/2}\int_0^\infty u^{\delta-1}(1+u^2)^{-k/2+1/4}\mathrm{d}u.$$

Since $\delta > 0$, the integral converges so that

$$c_p(1/2+\delta) \ll p^{k-1} \sum_{r=0}^{\infty} (r^2+4p)^{-k/2+1/4+\delta/2+(3/16+\varepsilon)(1-2\delta)}$$

for any $\delta > 0$. The series is dominated by the integral

$$\int_0^{\infty} (x^2+4p)^{-v} = \frac{\Gamma(v-1/2)\Gamma(1/2)}{2\Gamma(v)}(4p)^{1/2-v},$$
$$v = k/2 - 1/4 - \delta/2 - (3/16+\varepsilon)(1-2\delta)$$

which easily derived by contour integration (or see [GR, p. 295, formula 2]). Setting $k = 12$, we deduce

$$\tau(p) = O(p^{6-1/16+\varepsilon}),$$

as desired.

Remark. If we use the Lindelöf hypothesis instead of the Burgess-Heath-Brown estimate, we get $\tau(p) = O(p^{11/2+1/4+\varepsilon})$. A similar result holds for any cusp form of weight $k < 24$ or $k = 26$ for the full modular group because in these cases, the dimension of the space of cusp forms is $\leqslant 1$.

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