Some Remarks on Iterated Maps of Natural Numbers

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The iterates of maps $f : \mathbb{N} \to \mathbb{N}$ given as a function of the digits of the number written in a fixed base b are dealt with here. For such maps, the iterates end up in a finite collection of cycles. The number and length of such cycles have arithmetic significance.

1. Introduction

In preparing to write a congratulatory note to John Meisel on his 89th birthday, one of the authors looked at the volume *The Penguin Dictionary of Curious and Interesting Numbers* by D Wells (1998) that Meisel had given her earlier. For the number 89, it was noted that among other properties, 'Add the squares of the digits of any number: repeat this process, and eventually the number [sequence] either sticks to 1, or goes round this cycle: 89-145-42-20-4-16-37-58-89 ... .' It was this property that set the authors on the study that resulted in this article.

The first appearance of this intriguing phenomenon is in an old problem of Steinhaus [1] which asks the following. Take any natural number and add the square of its digits (in base 10). Iterating this procedure with the new numbers obtained, show that either this procedure leads to the number 1 or it will lead to the following recurrent cycle:

$$89, 145, 42, 20, 4, 16, 37, 58, 89, \ldots.$$

This problem is not difficult to solve. However, the problem can be placed in a wider context, as we do, and then new questions arise about the iterates of such maps and some of these questions are still unsolved.

What is fascinating is that something from recreational mathematics can lead to some profound questions in dynamical systems of natural numbers. We explore this here.

It is unlikely that the problem originates from Steinhaus. It is equally unlikely that it comes from Reg Allenby's daughter as described on page 234 of [2] who describes numbers that terminate at 1 as "happy numbers". It is conjectured that about 1/7 of the set of natural numbers are 'happy' though it is not clear if such numbers even have a density. Can one obtain estimates for upper and lower densities? Seemingly, more conjectures abound. What about the occurrence of consecutive happy numbers? Are there infinitely many such pairs? How large are the gaps between happy numbers? El-Sedy and Siksek [3] showed that there exist sequences of consecutive happy numbers of arbitrary length. This was subsequently generalized by Pan [4] who studied the analogous question replacing base 10 with base b. Any number theorist knows that such questions can be asked ad infinitum but the value of these questions emanates from its fecundity to generate new mathematical concepts and reveal interconnections with other parts of mathematics.

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The novelty of the question given here is its relation to dynamical systems and number theory. This topic seems to have been re-discovered many times before by many authors as the listing of our references reveals. There are at least five papers by Grundman and Teeple [5–9] on this problem.

Let $\mathbb{N}$ be the set of natural numbers and $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be an arbitrary map. For each natural number n written in base b:

El-Sedy and Siksek showed that there exist sequences of consecutive happy numbers of arbitrary length.

$$n = a_0 + a_1 b + \cdots + a_{k-1} b^{k-1}$$

with k digits (in base b), define the map

$$\phi(n) = f(a_0) + f(a_1) + \cdots + f(a_{k-1}).$$

Our first result is:

Theorem 1. *There is a finite set S such that for every natural number n, there is an r such that the rth iterate $\phi^r(n)$ lies in S.*

Remark. Here $\phi^r = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{r}$ times.

Proof. Since $0 \le a_i \le b-1$, we can set

$$M = \max_{0 \le a < b} f(a).$$

Then, $\phi(n) \le kM$. Now,

$$k - 1 \le \frac{\log n}{\log b} < k$$

so that

$$k = \left[\frac{\log n}{\log b}\right] + 1.$$

We easily see that

$$\phi(n) \le kM = \left(\left[\frac{\log n}{\log b}\right] + 1\right) M < n$$

if $n \ge n_0$ (say). Thus, for n sufficiently large, $\phi(n) < n$. In particular, there is a k_0 such that if n has more than k_0 digits, then $n \ge n_0$ so that $\phi(n) < n$. Let

$$S = \{n : n \,\text{has} \le k_0 \,\text{digits in base}\, b\} \cup \{\phi(n) : n \le n_0\}.$$

Then, it is easily seen that S satisfies the requirements of the theorem. □

2. The Herzberg Maps

For each natural number m, we consider the map

$$\phi_m(n) = a_0^m + a_1^m + \cdots + a_{k-1}^m.$$

For these maps (first brought to the attention of the second author by the first author, and hence the name), termed Herzberg maps, more precise inequalities and descriptions of S can be derived. Clearly,

$$\phi_m(n) \le (b-1)^m k.$$

We begin with the following lemma.

Lemma 2. *If $k \ge m+2$, then*

$$(b-1)^m k \le b^{k-1}\,.$$

Proof. Consider the function

$$F(x) = b^{x-1} - x(b-1)^m.$$

Note that

$$F'(x) = (\log b) b^{x-1} - (b-1)^m > 0$$

if $x \ge m+1$. In other words, $F(x)$ is increasing for $x \ge m+1$. Thus,

$$F(x) \ge F(m+2) = b^{m+1} - (m+2)(b-1)^m > 0$$

since

$$\begin{aligned} b^{m+1} &= (b-1+1)^{m+1} \\ &= (b-1)^{m+1} + (m+1)(b-1)^m + \cdots \\ &> (b-1)^{m+1} + (m+1)(b-1)^m \\ &= (m+2)(b-1)^m. \end{aligned}$$

In other words, $(b-1)^m k \le b^{k-1}$ *if $k \ge m+2$*, as claimed. This proves:

Theorem 2. *If $k \ge m+2$, then*

$$\phi_m(n) < n,$$

that is, if n has more than $m+2$ digits, then $\phi_m(n) < n$. Thus, for the Herzberg maps, ϕ_m, we can take $S = S_m$

to be the set of numbers with less than or equal to $m+1$ digits.

Remark. Theorem 2 reduces the determination of the possible values of the iterations of ϕ_m to a finite calculation. In the special case $b = 10$ and $m = 2$, corresponding to the original Steinhaus problem, we only need to calculate the orbits of ϕ_2 for all numbers less than 1000.

If b is small (e.g., $b = 2$ or 3), this is manageable since we need to check in general b^3 numbers. Thus, for $b = 2$, we need to check the orbits of

$$1, 10, 11, 100, 101, 110, 111$$

and in each case we see that the iterations terminate at 1. In fact, for $b = 2$, we have a strengthening of Theorem 2.

Theorem 3. *If $b = 2$, then $\phi_m^j(n) = 1$ for all n and j sufficiently large (depending on n and m).*

Proof. In base 2, any digit is either 0 or 1. Thus, if

$$\phi_m(n) = \sum_i a_i^m$$

with

$$n = \sum_i a_i 2^i,$$

then

$$\phi_m(n) = \sum_i a_i < \sum_i a_i 2^i = n$$

for $n > 1$ so that the iterates form a descending sequence which eventually terminates at 1.

For $b = 3$ and $m = 2$, this is also manageable. We need to check 27 numbers. A quick check shows that only two possibilities arise: ϕ_2 has 1, 5, and 8 as fixed points and the cycle $\{2, 4\}$ are the only possible terminal points of the iterates. Now

$$\begin{aligned} n &= \textstyle\sum_i a_i 3^i \equiv \sum_i a_i (\mathrm{mod}\, 2) \\ &\equiv \textstyle\sum_i a_i^2 = \phi_2(n) (\mathrm{mod}\, 2) \end{aligned}$$

because $a^2 \equiv a(\bmod 2)$. Thus, iterating we get,

$$\phi_2^k(n) \equiv n(\bmod 2)$$

so that we deduce:

Theorem 4. *If $b = 3$, then $\phi_2(n) \equiv n(mod\ 2)$. In particular, the terminal point is odd if and only if n is odd.*

The previous argument works for any odd base b. Indeed, $a^m \equiv a(\bmod\ 2)$ and we have

$$\phi_m(n) = \sum_i a_i^m \equiv \sum_i a_i \equiv \sum_i a_i b^i (\bmod 2).$$

In other words, $\phi_m(n)$ and n have the same parity. This leads to:

Theorem 5. *Let the base b be odd. If the terminal point of the iterates of $\phi_m(n)$ is 1, then n is odd. In other words, the density of numbers with 1 as the terminal point is at most 1/2.*

This theorem can be extended to give the following.

Theorem 6. *Let p be a prime number and suppose that $b \equiv 1 \ (\bmod\ p)$. If m is a power of p, then,*

$$n \equiv \phi_m(n)(\bmod p).$$

In particular, the density of numbers with 1 as the terminal point is at most $1/p$.

Proof. Since $b \equiv 1(\bmod\ p)$,

$$n \equiv \sum_i a_i (\bmod p).$$

On the other hand, by Fermat's little theorem, we have $a^p \equiv a \ (\bmod\ p)$ and by induction, $a^{p^r} \equiv a \ (\bmod\ p)$ for any $r \geq 1$. so that

$$\phi_m(n) = \sum_i a_i^m \equiv \sum_i a_i \equiv \sum_i a_i b^i \equiv n(\bmod p),$$

For a base b congruent to 1 modulo a prime p, then the density of numbers with 1 as the terminal number is at the most $1/p$.

which completes the proof since the final assertion follows immediately on noting that the density of numbers $\equiv 1 \pmod{p}$ is equal to $1/p$. $\square$

3. Fixed Points of ϕ_2

The eventual cycles of numbers and their lengths for the Herzberg maps seem to be of some interest. Cycles of length 1 correspond to fixed points of these maps. In the case of ϕ_2, the fixed points are easily determined as follows. We will show that if $\phi_2(n) = n$, then $n < b^2$. This helps us to reduce the search for fixed points.

Theorem 7. *If $\phi_2(n) = n$, then $n < b^2$.*

Proof. Write

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0 = a_k^2 + a_{k-1}^2 + \cdots + a_1^2 + a_0^2$$

so that

$$a_k(b^k - a_k) + \cdots a_1(b - a_1) = a_0^2 - a_0.$$

The left-hand side is a sum of non-negative terms as $b > a_i$ and so it is greater than or equal to $a_k(b^k - a_k)$ as $a_k \neq 0$. But $a_0^2 - a_0 = a_0(a_0 - 1) < b(b-1) = b^2 - b$. Also,

$$a_k(b^k - a_k) \geq b^k - a_k \geq b^2 - (b-1) = b^2 - b + 1$$

which is a contradiction. Thus, any fixed point n satisfies $n < b^2$. $\square$

This last result allows us to connect this problem to the classical number theoretic problem of representing a number as a sum of two squares. Indeed, from Theorem 7, if we want to determine the fixed points of ϕ_2, we need only consider numbers with two digits. Thus, $n = a_0 + a_1 b$ and $\phi_2(n) = n$, if and only if

$$a_0^2 + a_1^2 = a_0 + a_1 b\,.$$

This is equivalent to

$$a_0^2 - a_0 + a_1^2 - a_1 b = 0$$

which means

$$\left(a_0 - \frac{1}{2}\right)^2 + (a_1 - b2)^2 = \frac{1}{4} + \frac{b^2}{4},$$

that is, if and only if

$$(2a_0 - 1)^2 + (2a_1 - b)^2 = 1 + b^2.$$

In other words, to find the fixed points of ϕ_2, we need only find all the representations of $1 + b^2$ as a sum of two squares and from these representations determine the fixed points. For example, if $1 + b^2 = p$ is prime, then as there is only one way to write a prime congruent to 1 (mod 4) as a sum of two positive squares, we must have $a_0 = 1$ and $a_1 = 0$ (since $a_1 = b$ is ruled out because the digits are less than b) and this corresponds to $n = 1$ as being the only fixed point. In particular, for $b = 10$, 1 is the only fixed point since $101 = 1 + 10^2$ is a prime.

In other words, to find the fixed points of ϕ_2, we need only find all the representations of $1+b^2$ as a sum of two squares and from these representations determine the fixed points.

This pretty result has been re-discovered by many. For example, Beardon [10] showed this in 1998, although there were others before him who also discovered it.

4. The Case $k = 3$

Iseki [11, 12] obtained the complete list of cycles for the case $k = 3$ and $b = 10$. The cycles are of length 3 starting with 55 or 160, or length 2 starting with 136 or 919 or length 1 with 1, 153, 370 or 407. This was obtained through extensive computer calculations.

5. Open Questions

As can be seen from the case $k = 3$, the questions of how many cycles there are and what their lengths are, are not transparent. However, even for $k = 2$ and $b = 10$, there seem to be several unresolved aspects. These questions have also risen in several later studies [13, 14]. For instance, for $k = 2$, (as noted earlier), Guy [2] calls numbers that terminate at 1 'happy numbers' and asks "what is the density of such numbers". It is unclear if the density exists, though there have been some recent papers in this context (see for example [4]).

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