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Higher convolutions of Ramanujan sums

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We derive a limit formula for higher convolutions of Ramanujan sums, generalizing an old result of Carmichael. We then apply this in conjunction with the general theory of arithmetical functions of several variables to give a heuristic derivation of the Hardy-Littlewood formula for the number of prime k-tuplets less than x.

1. Introduction

An arithmetical function (of a single variable) is a map $f : \mathbb{N} \to \mathbb{C}$. In number theory and algebra, there is a plethora of such functions. For example, the number of divisors of a natural number n (denoted by $d(n)$), or the Euler totient function $\phi(n)$ or the von Mangoldt function $\Lambda(n)$ that occurs in prime number theory, are all instances of

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such functions. The study of such functions and their associated Dirichlet series forms a chapter of number theory often called multiplicative number theory.

A similar theory of arithmetical functions of several variables was initiated by Vaidyanathswamy [16] in 1927. Apart from sporadic and isolated results, no formal theory has emerged and it seems timely to delineate such a theory. Several expositions will assist us in developing the theory such as the one by Tòth [15]. We will give a synoptic introduction to this theory in section 2 below.

In this note, we will derive a generalization of a limit theorem of Carmichael involving Ramanujan sums which naturally leads to certain arithmetical functions of several variables. Then, we apply the general theory to give a heuristic derivation of the Hardy and Littlewood prime k-tuple conjecture formulated by them using the more complicated circle method. We first recall their conjecture.

In 1923, Hardy and Littlewood [5] generalized the celebrated twin prime conjecture and formulated what is now called the prime k-tuple conjecture, which is the following. Suppose that $d_1, ..., d_k$ are distinct integers, and let $b(p)$ be the number of distinct residue classes (mod p) represented by the d_i. If $b(p) < p$ for every prime p, the prime k-tuple conjecture asserts that the number of $n \leq x$ such that **all** the k numbers $n + d_i$ are prime for $1 \leq i \leq k$ is asymptotic to

$$\mathfrak{S}(d_1, ..., d_k) \frac{x}{(\log x)^k},$$

where

$$\mathfrak{S}(d_1, ..., d_k) = \prod_p \left(1 - \frac{b(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-k}, \tag{1}$$

and the product is over all primes p. Hardy and Littlewood formulated their conjecture using the intuition provided by the circle method and essentially ignoring the contribution from the so-called minor arcs emanating from the technique and focusing only on the major arcs. Though the idea is simple, the analysis of the major arcs was complicated and delicate. In 1999, Gadiyar and Padma [3] discovered an elementary heuristic to derive the case $k = 2$ (or the generalized twin prime conjecture) using a simple orthogonality principle for Ramanujan sums originally discovered by Carmichael [1]. In a recent paper, the authors along with Chaubey [2] have generalized the approach of Gadiyar and Padma and gave a heuristic derivation in the case $k = 3$. This led to the discovery of some new variants of Ramanujan sums that are of independent interest. Unfortunately, the generalization of this approach for the case when $k > 3$ leads to a calculation of exponential sums of several variables, which is not easy to solve. Therefore, in this paper, we adopt a different method which naturally leads to the study of certain arithmetical functions of several variables.

First, let us recall the Ramanujan sum:

$$c_q(n) = \sum_{(a,q)=1} e^{2\pi i a n/q},$$

originally defined by Ramanujan in [10] in 1918. One can easily find a 'closed' formula for $c_q(n)$ using the Möbius function. Indeed, we have

$$c_q(n) = \sum_{\substack{d|q \\ d|n}} d\mu(q/d). \tag{2}$$

We refer to [6,8] and [10] for the general properties of Ramanujan sums. Ramanujan [10] obtained the trigonometric series representations of normalized arithmetical functions of n in the form of an infinite series

$$\sum_{q=1}^{\infty} a_q c_q(n). \tag{3}$$

These series are now called the Ramanujan-Fourier series. The existence and convergence properties of these series are subjects that generate significant interest. A comprehensive review paper by Lucht [7] discusses the Ramanujan expansion of arithmetical functions. Moreover, notable monographs in this direction include the works of [11] and [14].

2. A synoptic view of arithmetical functions of several variables

An arithmetical function of several variables is a map $f : \mathbb{N}^k \to \mathbb{C}$. We will use vector notation as much as possible. Thus $\underline{n}$ will denote the k-tuple $(n_1, \cdots, n_k)$. We will say the vector $\underline{d}$ divides $\underline{n}$ (and write $\underline{d}|\underline{n}$) if $d_i|n_i$ for $1 \leq i \leq k$. The constant function $\underline{1}$ is simply the function that assigns the value 1 for every k-tuple. We will write $\underline{n}/\underline{d}$ to mean the vector $(n_1/d_1, ..., n_k/d_k)$.

We define the Möbius function $\underline{\mu}$ by

$$\underline{\mu}(\underline{n}) := \mu(n_1) \cdots \mu(n_k),$$

where μ is the classical Möbius function. We then have the generalization of the Möbius inversion formula:

$$f(\underline{n}) = \sum_{\underline{d}|\underline{n}} g(\underline{d}) \iff g(\underline{n}) = \sum_{\underline{d}|\underline{n}} \underline{\mu}(\underline{d}) f(\underline{n}/\underline{d}).$$

There are several ways to generalize the notion of a multiplicative function of a single variable to the several variable context. In 1931, Vaidyanathaswamy [16] was the first to give the definition that is suitable for our purposes. Selberg [13] seems to have rediscovered this definition much later in 1977 in his paper dealing with extensions of the large sieve.

We say a function f is multiplicative if

$$f(m_1, ..., m_k)f(n_1, ..., n_k) = f(m_1n_1, ..., m_kn_k)$$

provided $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. With this definition, it is **not true** that if we fix one component, n_1 (say), then $f(n_1, ..., n_k)$ is a multiplicative function in the remaining variables $n_2, ..., n_k$. (Selberg says otherwise on pages 233-234 in his paper [13] and as he does not use this, the results of his paper are unaffected.) For instance, the Ramanujan sum $c_q(n)$ is a multiplicative function of q for fixed n but is not a multiplicative function of n for fixed q. However, $c_q(n)$ is a multiplicative function of two variables q, n as we have defined it above using Vaidyanathaswamy's definition. In particular, $\underline{\mu}$ is multiplicative and generally, a multiplicative function f is completely determined by its values $f(p^{v_1}, ..., p^{v_k})$ for every prime p and every tuple $(v_1, ..., v_k) \in \mathbb{N}^k$.

It is not hard to see that if f and g are multiplicative, then so is their Dirichet convolution $f \star g$ defined as

$$(f \star g)(n) = \sum_{\underline{d}|\underline{n}} f(\underline{d})g(\underline{n}/\underline{d}).$$

For multiplicative functions f, we can introduce a formal Dirichlet series of several variables along with an Euler product:

$$\sum_{\underline{n}=\underline{1}}^{\infty} \frac{f(n_1, ..., n_k)}{n_1^{s_1} \cdots n_k^{s_k}} = \prod_p \left(\sum_{v_1,...,v_k=0}^{\infty} \frac{f(p^{v_1}, ..., p^{v_k})}{p^{v_1 s_1} \cdots p^{v_k s_k}} \right).$$

3. Generalized Chinese remainder theorem

We will use the following variant of the classical Chinese remainder theorem. The familiar version is often stated when the $d_1, ..., d_k$ are pairwise coprime. It is a simple exercise to derive this general version from the classical version (see for example, the inductive proof on page 155 of [12]).

Lemma 3.1. *For a fixed set $T = \{a_1, \cdots, a_k\}$ and $d_1, \cdots, d_k \in \mathbb{Z}$, the system*

$$\begin{aligned} x &\equiv a_1 \mod d_1 \\ &\vdots \\ x &\equiv a_k \mod d_k \end{aligned} \tag{4}$$

has a solution if and only if $(d_i, d_j)|(a_i - a_j)$ for all $1 \le i, j \le k$. When the solution exists, it is unique modulo $[d_1, \cdots, d_k]$.

Proof. Our proof is direct and more conceptual than the one in [12]. For a prime p, let $v_p(n)$ be the largest power of p dividing n. Then, the system of congruences (4) is equivalent to $x \equiv a_i \pmod{p^{v_p(d_i)}}$ for $1 \le i \le k$ and all primes p dividing the lcm $[d_1, ..., d_k]$. It therefore suffices to prove the theorem when all the d_i are the powers of a single prime p. The result is now self-evident since the existence of a solution implies that $(d_i, d_j)|(a_i - a_j)$ for all $1 \le i, j \le k$. For the converse, the condition that $(d_i, d_j)|(a_i - a_j)$ for all $1 \le i, j \le k$ implies the compatibility of the a_i. That is, if $v_p(d_i) \le v_p(d_j)$, then a_j is indeed a "lift" $\pmod{d_j}$ of a_i as required. □

From now on, we will fix T and define a function

$$g(d_1, \ldots, d_k) := \begin{cases} 1 & \text{if (4) has a solution,} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

4. Higher convolutions of Ramanujan sums

In 1932, Carmichael [1] discovered the following 'orthogonality' property of Ramanujan sums: for $h \neq 0$,

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} c_r(n) c_s(n+h) = \begin{cases} c_r(h) & \text{if } r = s \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

We will generalize this limit theorem in the following way. Let $T = \{a_1, a_2, ..., a_k\}$ be a given multiset of integers. Then, the limit

$$f(q_1, ..., q_k) := \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} c_{q_1}(n + a_1) \cdots c_{q_k}(n + a_k),$$

exists and can be evaluated as follows. From (2), we have

$$f(q_1, ..., q_k) = \lim_{x\to\infty} \frac{1}{x} \sum_{d_1|q_1,...,d_k|q_k} d_1 \mu\left(\frac{q_1}{d_1}\right) \cdots d_k \mu\left(\frac{q_k}{d_k}\right) \sum_{\substack{n\le x \\ d_1|a_1+n,\cdots,d_k|a_k+n}} 1.$$

Therefore, from (5) we have

$$f(q_1, ..., q_k) := \sum_{d_1|q_1,...,d_k|q_k} d_1 \mu\left(\frac{q_1}{d_1}\right) \cdots d_k \mu\left(\frac{q_k}{d_k}\right) \frac{g(d_1, ..., d_k)}{[d_1, ..., d_k]}. \tag{7}$$

Since $g(d_1, ..., d_k)$ is multiplicative, we see that $f(n_1, ..., n_k)$ is multiplicative. This proves the following generalization of Carmichael's theorem.

Theorem 4.1. *For fixed integers* $a_1, \cdots, a_k$ *and* $q_1, \cdots, q_k$*, we have*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} c_{q_1}(n+a_1)\cdots c_{q_k}(n+a_k) = \sum_{d_1|q_1,\ldots,d_k|q_k} d_1\mu\left(\frac{q_1}{d_1}\right)\cdots d_k\mu\left(\frac{q_k}{d_k}\right)\frac{g(d_1,\ldots,d_k)}{[d_1,\ldots,d_k]}.$$

Since the function on the right-hand side of the above theorem is a multiplicative function, it suffices to determine the values $f(p^{v_1},\ldots,p^{v_k})$ for a fixed prime p. For our application, we need this when all the v_i are less than or equal to 1. We will derive the required formula in the next section.

It is worth remarking that in the case $k = 2$, our Theorem 4.1 agrees with Carmichael's theorem. Indeed, to verify this, we need to compute explicitly $f(q_1, q_2)$ and ascertain its identity with Carmichael's limit. That is, we must check $f(q_1, q_2) = 0$ if $q_1 \neq q_2$ and $c_q(h)$ when $q_1 = q_2 = q$. By multiplicativity, it suffices to determine $f(p^a, p^b)$ for a fixed prime p. Without any loss of generality, we may suppose that $a \leq b$. The sum (7) has only four terms corresponding to $d_1 = p^a$ or p^{a-1} and $d_2 = p^b$ or p^{b-1}. In the case $a \leq b-1$, the summation is easily checked to be zero. In the case $a = b$, the summation is $p^a - p^{a-1} = c_{p^a}(h)$ since $p^a|h$ by the compatibility condition of Lemma 3.1 to ensure a solution.

5. Explicit evaluation of $f(p^{v_1},\ldots,p^{v_k})$

We define an equivalence relation on $\{1, 2, \ldots, k\}$ using T. We say $i \sim j$ if and only if $a_i \equiv a_j \pmod p$. This partitions T into equivalence classes C_i. Let $b(p)$ be the number of equivalence classes. Note that this induces an equivalence relation on any subset S of $\{1, 2, \ldots, k\}$ and the corresponding equivalence classes for S are simply $S \cap C_i$ (some of which can be empty).

Lemma 5.1. *For* $0 \leq v_i \leq 1$ *for* $1 \leq i \leq k$*, we have*

$$f(p^{v_1},\ldots,p^{v_k}) = (-1)^{|S|} + \frac{(-1)^{|S|}}{p}\sum_{C_i}[(1-p)^{|C_i\cap S|} - 1]$$

where $S = \{i : v_i = 1\}$.

Proof. As remarked earlier, the equivalence relation on T induces an equivalence relation on S. From (7), we see that in the sum for $f(p^{v_1},\ldots,p^{v_k})$, the contribution from $d_1 = d_2 = \cdots = d_k = 1$ is $(-1)^{|S|}$. For the remaining tuples of divisors $(d_1,\ldots,d_k)$, we must have $d_i = p$ for some $i \in S$. If $d_j = p$ for some other $j \neq i$, then we must have j equivalent to i by the definition of our equivalence relation. In other words, the remaining sum can be re-written as

$$\sum_{C_i}\sum_{j=1}^{|C_i\cap S|}\binom{|C_i\cap S|}{j}p^{j-1}(-1)^{|S|-j} = \frac{(-1)^{|S|}}{p}\sum_{C_i}[(1-p)^{|C_i\cap S|} - 1]$$

which completes the proof. □

Theorem 5.2. *Let* $T \pmod p$ *have size* $b(p)$*. Then,*

$$\sum_{v_1,\dots,v_k\geq 0} \frac{\mu(p^{v_1})\cdots\mu(p^{v_k})}{\phi(p^{v_1})\cdots\phi(p^{v_k})} f(p^{v_1},...,p^{v_k}) = \left(1-\frac{b(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k}$$

Proof. To evaluate the sum on the left hand side, we need only consider the terms with $v_i \leq 1$ for all $1 \leq i \leq k$ because the Möbius function vanishes otherwise. We insert our formula for $f(p^{v_1},...,p^{v_k})$ from Lemma 5.1 into the sum to get

$$\sum_{v_1,\dots,v_k\geq 0} \frac{\mu(p^{v_1})\cdots\mu(p^{v_k})}{\phi(p^{v_1})\cdots\phi(p^{v_k})} \left\{(-1)^{|S|} + \frac{(-1)^{|S|}}{p}\sum_{C_i}[(1-p)^{|C_i\cap S|}-1]\right\},$$

where $S = \{i : v_i = 1\}$ (as before). The first part of the sum is easily evaluated:

$$\sum_{v_1,\dots,v_k\geq 0} \frac{\mu(p^{v_1})\cdots\mu(p^{v_k})}{\phi(p^{v_1})\cdots\phi(p^{v_k})}(-1)^{|S|} = \sum_{j=0}^{k}\binom{k}{j}\frac{1}{(p-1)^j} = \left(1+\frac{1}{p-1}\right)^k. \tag{8}$$

The second part of the sum is a bit more delicate. Let $[k]$ denote the set $\{1, 2, ..., k\}$. Since the product of the Möbius functions is $(-1)^{|S|}$ and the product of the ϕ functions is $(p-1)^{|S|}$, we get

$$\frac{1}{p}\sum_{\emptyset\neq S\subseteq[k]} \frac{1}{(p-1)^{|S|}}\sum_{C_i}[(1-p)^{|C_i\cap S|}-1].$$

We interchange the sums to get

$$\frac{1}{p}\sum_{C_i}\sum_{\emptyset\neq S\subseteq[k]} \frac{1}{(p-1)^{|S|}}[(1-p)^{|C_i\cap S|}-1].$$

We examine the inner sum. Writing $A = C_i \cap S$ we see that $S = A \sqcup B$ (where $\sqcup$ denotes disjoint union) and $B \subseteq [k]\backslash C_i$. Since $|S| = |A| + |B|$, the sum becomes

$$\frac{1}{p}\sum_{C_i}\sum_{A\subseteq C_i}\frac{(1-p)^{|A|}-1}{(p-1)^{|A|}}\sum_{B\subseteq[k]\backslash C_i}\frac{1}{(p-1)^{|B|}}.$$

The innermost sum is equal to

$$\left(1+\frac{1}{p-1}\right)^{k-|C_i|}.$$

Now

$$\sum_{A\subseteq C_i} \frac{(1-p)^{|A|}-1}{(p-1)^{|A|}} = -\left(1+\frac{1}{p-1}\right)^{|C_i|},$$

because

$$\sum_{A\subseteq C_i} \frac{(1-p)^{|A|}}{(p-1)^{|A|}} = \sum_{A\subseteq C_i} (-1)^{|A|} = 0.$$

Putting everything together gives

$$-\frac{1}{p}\sum_{C_i}\left(1+\frac{1}{p-1}\right)^{|C_i|}\left(1+\frac{1}{p-1}\right)^{k-|C_i|} = -\frac{b(p)}{p}\left(1+\frac{1}{p-1}\right)^{k}.$$

Combining this with the first part (8) gives the desired result:

$$\left(1+\frac{1}{p-1}\right)^{k}\left(1-\frac{b(p)}{p}\right) = \left(1-\frac{b(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k}. \quad \square$$

6. A heuristic derivation of the Hardy-Littlewood k-tuple conjecture

We can now combine the above discussion and give the promised heuristic derivation of the Hardy-Littlewood prime k-tuple conjecture. By partial summation, the conjecture is easily seen to be equivalent to

$$\sum_{n\le x} \Lambda(n+a_1)\cdots\Lambda(n+a_k) \sim x\prod_p \left(1-\frac{b(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k}, \tag{9}$$

where $b(p)$ is the size of the image of $T \bmod p$.

Our objective is to present a heuristic proof of (9) by employing the convolution of Ramanujan sums. First, we observe that

$$\Upsilon := \sum_{n\le x} \Lambda(n+a_1)\cdots\Lambda(n+a_k) \sim \sum_{n\le x} \frac{\phi(n+a_1)}{n+a_1}\cdots\frac{\phi(n+a_k)}{n+a_k}\Lambda(n+a_1)\cdots\Lambda(n+a_k).$$

To see this, we need only note that the sum on the left hand side is negligible if $n \le x^{1-\epsilon}$ for any $\epsilon > 0$. Indeed,

$$\sum_{n\le x^{1-\epsilon}} \Lambda(n+a_1)\cdots\Lambda(n+a_k) \ll x^{1-\epsilon}(\log x)^k \ll x^{1-\epsilon'},$$

for any ϵ' with $0 < \epsilon' < \epsilon < 1$. Similarly, the sum on the right hand side can also be restricted to $x^{1-\epsilon} \le n \le x$ and in this interval, we have

$$\frac{\phi(n+a_i)}{n+a_i}\Lambda(n+a_i) \sim \Lambda(n+a_i).$$

This modification enables us to use Hardy's formula (see for example, [9]):

$$\frac{\phi(n)\Lambda(n)}{n} = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n), \tag{10}$$

where the summation is over squarefree q. Next, inserting the Ramanujan-Fourier series of Hardy from equation (10) and ignoring issues of convergence, we have upon using Theorem 4.1,

$$\frac{\Upsilon}{x} \sim \sum_{q_1,\cdots,q_k=1}^{\infty} \frac{\mu(q_1)\cdots\mu(q_k)}{\phi(q_1)\cdots\phi(q_k)} f(q_1,\cdots,q_k).$$

Therefore, the Hardy-Littlewood constant is equal to

$$\sum_{q_1,\ldots,q_k=1}^{\infty} \frac{\mu(q_1)\cdots\mu(q_k)}{\phi(q_1)\cdots\phi(q_k)} f(q_1,\ldots,q_k). \tag{11}$$

We want to show that this agrees with the classical evaluation of this constant as

$$\prod_p \left(1 - \frac{b(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-k},$$

where $b(p)$ is the size of the image of $T \pmod{p}$.

By multiplicativity, the series in (11) can be written as the Euler product:

$$\prod_p \left(\sum_{v_1,\ldots,v_k \geq 0} \frac{\mu(p^{v_1})\cdots\mu(p^{v_k})}{\phi(p^{v_1})\cdots\phi(p^{v_k})} f(p^{v_1},\ldots,p^{v_k}) \right).$$

We now examine the p-Euler factor and evaluate explicitly $f(p^{v_1},\ldots,p^{v_k})$ for $0 \leq v_i \leq 1$ for $1 \leq i \leq k$. Let us henceforth fix p, then from Theorem 5.2, we obtain the required result.

7. Concluding remarks

It would be of some value to develop a theory of truncated Ramanujan expansions in the multi-variable context. This was initiated in [4] in the single variable case. Such a line of research will have applications in the study of general convolution sums, perhaps not immediately to convolutions of the von Mangoldt function as it occurs in the prime k-tuple conjecture.

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Data availability

No data was used for the research described in the article.

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