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BERTRAND'S POSTULATE FOR NUMBER FIELDS

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Abstract. Consider an algebraic number field, K, and its ring of integers, $\mathcal{O}_K$. There exists a smallest $B_K > 1$ such that for any $x > 1$ we can find a prime ideal, $\mathfrak{p}$, in $\mathcal{O}_K$ with norm $N(\mathfrak{p})$ in the interval $[x, B_K x]$. This is a generalization of Bertrand's postulate to number fields, and in this paper we produce bounds on B_K in terms of the invariants of K from an effective prime ideal theorem due to Lagarias and Odlyzko (1977). We also show that a bound on B_K can be obtained from an asymptotic estimate for the number of ideals in $\mathcal{O}_K$ with norm less than x.

1. Introduction. Predating the prime number theorem, Bertrand's postulate was first put forward by Joseph Bertrand in 1845 and proved by Chebyshev in 1850. It states that, for any $x > 1$, a prime number can be found in the interval $[x, 2x]$. This is generally considered to be a much weaker result than the prime number theorem, as one can use the asymptotic behavior of the prime counting function, $\pi(x)$, to show that for any $A > 1$ there exists $x_A > 1$ such that for any $x > x_A$ we have $\pi(Ax) - \pi(x) > 0$, and so there is a prime number in the interval $[x, Ax]$. Thus, in principle, one can use the prime number theorem to bound x_A from above and find a lowest possible x_A by employing a finite search. Indeed, Betrand's postulate itself can be recovered using more precise upper and lower bounds for $\pi(x)$, like those due to Dusart [2] which arise from numerical verification of the Riemann hypothesis for the first $1.5 \cdot 10^9$ zeros.

Apart from historical interest, however, one of the main benefits of Bertrand's postulate is that it gives information about the distribution of primes when x is small. Furthermore, it accomplishes this without requiring information about the zeros of the Riemann zeta function. Indeed, Bertrand's postulate benefits from having many short and often elegant, elementary proofs [1, 9, 6].

One may similarly investigate a variant of Bertrand's postulate for the distribution of prime ideals in the ring of integers, $\mathcal{O}_K$, of an algebraic

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number field, K. That is, for general K, we ask if we can find B such that for all $x > 1$ there exists a prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ with norm $N(\mathfrak{p}) \in [x, Bx]$. Indeed, if such a B exists for any given number field we can define the *Bertrand constant*, B_K, to be the best such B,

$$B_K := \min\{B > 1 \mid \forall x > 1,\ \exists \mathfrak{p} \subseteq \mathcal{O}_K,\ N(\mathfrak{p}) \in [x, Bx]\}. \tag{1.1}$$

We can define B_K to be a minimum instead of an infimum, for if B_K is the infimum of the non-empty set then for any $x > 1$ we can find a prime ideal with norm in $[x, (B_K + \varepsilon)x]$ for any $\varepsilon > 0$. Since the norms of ideals are rational integers, we can take $\varepsilon > 0$ small enough so that there must be a prime ideal with norm in $[x, B_K x]$, and thus B_K is an element of the above set.

We know that B_K must exist due to the prime ideal theorem, first proven by Landau in 1903, which states that for $x > 1$,

$$\pi_K(x) \sim \frac{x}{\log x}, \tag{1.2}$$

where $\pi_K(x)$ counts the number of prime ideals in $\mathcal{O}_K$ with norm less than x. Generally, this theorem is given in the more effective form

$$\pi_K(x) = \mathrm{Li}(x) + O_K(xe^{-c_K\sqrt{\log x}}), \tag{1.3}$$

where $c_K > 0$ is dependent on K.

Our B_K and other similarly defined constants would allow us to produce analogues of Bertrand's postulate for the number field K, and though questions about the distribution of prime ideals are of great interest, it appears no attention has been paid to this problem outside of the case where $K = \mathbb{Q}$. We would like to investigate B_K for a non-trivial number field and the dependence it has on the invariants of the number field.

As with the proof of the prime number theorem, the prime ideal theorem is obtained by finding a zero-free region of the Dedekind zeta function, $\zeta_K(s)$, which is defined for $\Re(s) > 1$ as

$$\zeta_K(s) := \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_K \\ \mathfrak{a} \neq (0)}} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{c_K(n)}{n^s}, \tag{1.4}$$

where $\mathfrak{a}$ are the ideals in $\mathcal{O}_K$, and has a meromorphic continuation to all $s \in \mathbb{C}$. Like the Riemann zeta function, $\zeta_K(s)$ also has a functional equation and has only one pole at $s = 1$, which is simple with residue ρ_K. This residue is related to the invariants of K by the formula

$$\rho_K = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{|\Delta_K|}}. \tag{1.5}$$

Here Δ_K is the discriminant of K, h_K is its class number, R_K is its regulator, w_K is the number of roots of unity contained in K, and r_1 and r_2 are the numbers of real and complex embeddings of K, respectively. Moreover, let $d = [K : \mathbb{Q}]$.

In 1977, Lagarias and Odlyzko [5] were able to state effective versions of the Chebotarev density theorem, which generalizes the prime ideal theorem to prime ideals whose Frobenius automorphisms lie in fixed conjugacy classes. This in turn specializes to an effective version of the prime ideal theorem, which we formulate here:

THEOREM 1.1 (Lagarias, Odlyzko [5]). *If K is a number field, there exist effectively computable positive constants c_1 and c_2, independent of K, such that if $x \geq \exp(10d(\log|\Delta_K|)^2)$ then*

$$|\pi_K(x) - \operatorname{Li}(x) + (-1)^{\epsilon_K} \operatorname{Li}(x^\beta)| \leq c_1 x \exp\left(-c_2\sqrt{\frac{\log x}{d}}\right), \tag{1.6}$$

where $\operatorname{Li}(x^\beta)$ only occurs if there exists an exceptional real simple zero, β, of $\zeta_K(s)$ such that $1-(4\log|\Delta_K|)^{-1} < \beta < 1$. Also $\epsilon_K = 0$ or 1, depending on K.

If the Generalized Riemann Hypothesis (GRH) holds for $\zeta_K(s)$ then there exists an effectively computable positive absolute constant c_3 such that for $x > 2$,

$$|\pi_K(x) - \operatorname{Li}(x)| \leq c_3 x^{1/2} \log(|\Delta_K| x^d). \tag{1.7}$$

By using the above estimates, we can make an effort to determine when $\pi_K(Ax) - \pi_K(x) > 0$. The possible exceptional zero complicates what would otherwise be a fairly straightforward computation, and so we make use of an upper bound due to Stark [10], which itself depends on whether or not K is a normal field extension. The proof of the following theorem can be found in Section 2.

THEOREM 1.2. *Let $K \neq \mathbb{Q}$ be a finite field extension of $\mathbb{Q}$ such that there exists a tower of fields $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_m = K$ where each K_i is a finite normal extension of K_{i-1}. For any $A > 1$ there exists $c_A > 0$, dependent only on A, such that for $x > \exp(c_A d(\log|\Delta_K|)^2)$, there is a prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ with $N(\mathfrak{p}) \in [x, Ax]$.*

Now suppose that $K \neq \mathbb{Q}$ is a finite extension of $\mathbb{Q}$, but the tower of normal field extensions does not exist. Let $\log|\Delta_K| \gg d(\log d)^\alpha$ for $\alpha \in [0,1]$. For any $A > 1$ there exists $c_A > 0$, dependent only on A, such that for $x > \exp(c_A d(\log d)^{2-2\alpha}(\log|\Delta_K|)^2)$, there is a prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ with $N(\mathfrak{p}) \in [x, Ax]$.

Finally, suppose only that $K \neq \mathbb{Q}$ is a number field. If the GRH holds then for any $A > 0$ there exists c_A, dependent only on A, such that for

$$x > c_A(\log|\Delta_K| + d)^2 \log^4(\log|\Delta_K| + d)$$

there is a prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ with $N(\mathfrak{p}) \in [x, Ax]$.

REMARK 1.3. We note from Minkowski's bound and Stirling's approximation that it is always the case that $\log|\Delta_K| \gg d$. So by specifying that $\log|\Delta_K| \gg d(\log d)^\alpha$ for $\alpha \in [0,1]$ we are not excluding any cases.

Since increasing A means that we can decrease c_A, we can eventually get the following corollary extending Bertrand's postulate to number fields.

COROLLARY 1.4. *There exists an absolute constant c such that for any number field $K \neq \mathbb{Q}$, we have:*

(a) $B_K \le \exp(cd(\log|\Delta_K|)^2)$ *if there exists a tower of fields* $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_m = K$ *where each* K_i *is a finite normal extension of* K_{i-1}.

(b $B_K \le \exp(cd(\log d)^{2-2\alpha}(\log|\Delta_K|)^2)$ *if the tower of fields does not exist and when* $\log|\Delta_K| \gg d(\log d)^\alpha$ *for* $\alpha \in [0,1]$.

(c) $B_K \le c(\log|\Delta_K| + d)^2 \log^4(\log|\Delta_K| + d)$ *if the GRH holds,*

The proof of the effective prime ideal theorem is quite technically involved. Since some of our interest in Bertrand's postulate is due to the brevity and elegance of the proofs for it, one would hope that in generalizing Bertrand's postulate to number fields we could obtain a comparable result using a less elaborate argument and without requiring information about the zeros of $\zeta_K(s)$.

Let

$$f_1(x,K) := \sum_{n \le x} c_K(n) - \rho_K x, \tag{1.8}$$

where, as above, $c_K(n)$ are the coefficients of $\zeta_K(s)$, and ρ_K is the residue of $\zeta_K(s)$ at $s = 1$. That is, $f_1(x,K)$ is the error term for the number of ideals in $\mathcal{O}_K$ with norm less than x. Information about $f_1(x,K)$ alone is sufficient to obtain a generalized Bertrand postulate for a finite field extension K of $\mathbb{Q}$. We prove the following result in Section 3.

THEOREM 1.5. *Let K be a number field, as above. Suppose for fixed $0 < \alpha < 1$ that there exists some $\mathcal{C}_K > 0$, determined by the invariants of K, such that*

$$|f_1(x,K)| \le \mathcal{C}_K x^\alpha \tag{1.9}$$

for all $x \ge 1$. Then for any $x > 1$, there exists a prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ such that $N(\mathfrak{p}) \in [x, Ax]$ whenever

$$\log A \gg \frac{\mathcal{C}_K}{\rho_K}\frac{d+2}{1-\alpha} + d. \tag{1.10}$$

REMARK 1.6. It is common notation that $f(x) \ll g(x)$ indicates that $|f(x)| \le Cg(x)$ for particular values of x for some independent constant C. Throughout this work, however, we will say $f \ll g$ if there exists some constant C, which can be chosen independently of any of the variables or invariants that may define f and g, such that $|f| \leq C|g|$, unless otherwise specified. We say that $f \ll_d g$ if C has some dependence on d, and similarly for other variables. Our big-O notation reflects this as well. For example, we say $f = g + O_d(x)$ if $(f - g) \ll_d x$.

In 1972, by effectivizing Landau's theorem, Sunley [11] was able to derive a completely effective bound for $|f_1(x, K)|$.

Theorem 1.7 (Sunley [11]). *For $f_1(x, K)$ as in (1.8), we have*

$$|f_1(x,K)| < e^{56d+5}(d+1)^{5(d+1)/2}|\Delta_K|^{1/(d+1)}(\log^d |\Delta_K|)x^{(d-1)/(d+1)}. \tag{1.11}$$

Combining this with Theorem 1.5, and employing a theorem due to Friedman [3] which implies that

$$\frac{R_K}{w_K} \geq \frac{9}{100} \tag{1.12}$$

for all number fields K, we are able to produce the following corollary as a proof of concept.

Corollary 1.8. *Let K be a number field with Bertrand constant B_K. Then*

$$\log B_K \ll \frac{e^{\frac{5}{2}(d+9/5)\log(d+1)+(56-\log 2)d}}{h_K}|\Delta_K|^{1/2+1/(d+1)}\log^d|\Delta_K| + d, \tag{1.13}$$

where the implied constant is absolute.

This result is significantly worse than Corollary 1.4, but it has an advantage that the absolute constant is much easier to compute from the proof of Theorem 1.5. Indeed, we can do better than Sunley if we restrict our attention to just growth in the $|\Delta_K|$ aspect rather than attempting a hybrid bound in all the invariants of K.

With this in mind, we consider the following proposition.

Proposition 1.9. *For all $x \geq 1$ and small $\delta > 0$ such that $1/(3d) > \delta$ we have*

$$\sum_{n \leq x} c_K(n) = \rho_K x + O_{d,\delta}\big((\zeta_K(1 + \delta/2)|\Delta_K|^\delta + 1)x^{1-\delta/2}\big). \tag{1.14}$$

This asymptotic is obtained almost directly from the work of Kuo and Murty [4], albeit in such a way that the contribution from the discriminant is mitigated at the expense of growth in x. The proof of it can be found in the Appendix. While we know that $\zeta_K(1 + \delta/2) \leq \zeta^d(1 + \delta/2)$ and so $\zeta_K(1 + \delta/2)$ can be absorbed into the implied constant, we also know that

$$\lim_{\delta\to 0} \frac{\delta}{2}\zeta_K\left(1 + \frac{\delta}{2}\right) \to \rho_K, \tag{1.15}$$

though it is not obvious how small δ needs to be relative to the invariants of K for this to be a good approximation. Still, we heuristically expect $\zeta_K(1+\delta/2)$ to cancel the ρ_K in (1.10) in exchange for a large contribution due to δ, so we keep track of it. The obstacle to understanding the bound in the δ and d aspects is finding a good effective estimate for the implied constants for the bound $c_K(n) \ll_{\delta,d} n^\delta$.

Inputting (1.14) into (1.10), and again employing a theorem due to Friedman [3], we are able to produce the following corollary.

COROLLARY 1.10. *Let K be a number field with Bertrand constant B_K. Then*

$$B_K \le \exp\left(M_{d,\delta}\left(\frac{\zeta_K(1+\delta/2)|\Delta_K|^\delta}{\rho_K} + \frac{|\Delta_K|^{1/2}}{h_K}\right)\right) \tag{1.16}$$

for some constant $M_{\delta,d} > 0$ depending on d and δ where $1/(3d) > \delta > 0$.

The size of $|\Delta_K|^{1/2}/h_K$ depends on the existence of a Siegel zero, but in the case of totally complex number fields it is heuristically likely to grow like $\log|\Delta_K|$. If we could indeed let $\zeta_K(1+\delta)/\rho_K = O_\delta(1)$ then we would be left with the $|\Delta_K|^\delta$ term as a main term. So if we hope to match the result in Corollary 1.10 in the Δ_K aspect in any case, we would need to let $\delta = 2\log(\log|\Delta_K|)/\log|\Delta_K|$, but then we would be limited by the lack of effectiveness in the δ aspect.

Though the result of Theorem 1.5 and our current bounds for $|f_1(x,K)|$ are apparently worse than those that can be obtained from careful analysis of the effective prime ideal theorem, one would hope that they might be put to better use in certain special cases, such as for quadratic fields. This has some overlap with the older problem of Bertrand's postulate for primes in arithmetic progressions, and may be an avenue for further research. A thorough treatment of this topic can be found in Moree [7].

2. Bertrand's postulate from the prime ideal theorem

Proof of Theorem 1.2. First suppose that no exceptional zero exists. By Theorem 1.1 we only have to show that for any $A > 1$ there exists c_A, independent of $|\Delta_K|$ and d, such that

$$\pi(Ax) - \pi(x) > \mathrm{Li}(Ax) - \mathrm{Li}(x) - 2c_1 Ax \exp\left(-c_2\sqrt{\frac{\log x}{d}}\right) > 0 \tag{2.1}$$

for $x \ge \exp(c_A d(\log|\Delta_K|)^2)$. This is also the case if an exceptional zero exists and $\epsilon_K = 1$ since $\mathrm{Li}(x)$ is an increasing function for $x > 2$. From partial integration we have

$$\mathrm{Li}(Ax) - \mathrm{Li}(x) = \frac{Ax}{\log Ax} - \frac{x}{\log x} + \int_x^{Ax} \frac{dt}{(\log t)^2}, \tag{2.2}$$

so it suffices to show that

$$A\frac{\log x}{\log Ax} > 1 + 2Ac_1(\log x)\exp\left(-c_2\sqrt{\frac{\log x}{d}}\right). \tag{2.3}$$

Since only A determines how large x needs to be for $A\log x/\log Ax$ to be larger than 1, we need only show that we can make the exponential term sufficiently small by controlling the size of c_A independently of d and Δ_K. Indeed, let $x = \exp(c_A d(\log|\Delta_K|)^2)$. Then

$$2Ac_1(\log x)\exp\left(-c_2\sqrt{\frac{\log x}{d}}\right) = 2Ac_1(c_A d(\log|\Delta_K|)^2)|\Delta_K|^{-c_2\sqrt{c_A}}. \tag{2.4}$$

Taking advantage of Minkowski's bound and Stirling's approximation we can say that

$$d|\Delta_K|^{-1} \le \left(\frac{4}{\pi}\right)^d \frac{(d!)^2}{d^{2d-1}} \sim 2\pi\left(\frac{4}{\pi e^2}\right)^d \tag{2.5}$$

as d gets large. Thus the exponential term in (2.4) will decrease as c_A increases past a point that can be chosen independently of K, and so we can say that for $x > \exp(c_A d(\log|\Delta_K|)^2)$,

$$(\log x)\exp\left(-c_2\sqrt{\frac{\log x}{d}}\right) \le c_A d(\log|\Delta_K|)^2|\Delta_K|^{-c_2\sqrt{c_A}}. \tag{2.6}$$

We see the upper bound can be made uniform in K and also vanishes as $c_A \to \infty$, giving the proposition in this case.

For the case of the General Riemann Hypothesis, we follow the same reasoning but replace (1.7) with (1.6). So when

$$x > c_A(\log|\Delta_K| + d)^2\log^4(\log|\Delta_K| + d),$$

it suffices to observe, via substitution, that the term

$$\frac{\log x}{x}x^{1/2}\log(|\Delta_K|x^d) \tag{2.7}$$

can be made arbitrarily small by increasing c_A independently of K.

Now suppose an exceptional zero exists with $\beta > 1 - (4\log|\Delta|_K)^{-1}$ and $\epsilon_K = 0$. Then we need to show that

$$\mathrm{Li}(Ax) - \mathrm{Li}(x) - \big(\mathrm{Li}((Ax)^\beta) - \mathrm{Li}(x^\beta)\big) - 2c_1Ax\exp\left(-c_2\sqrt{\frac{\log x}{d}}\right) > 0 \tag{2.8}$$

for $x > \exp(c_A d(\log|\Delta_K|)^2)$. We deduce from Stark [10] that, if K is a normal extension, we can assume

$$1 - (4\log|\Delta|_K)^{-1} < \beta < 1 - c_4|\Delta_K|^{-1/d}, \tag{2.9}$$

if β exists, for some effectively computable positive constant c_4. Differentiation shows us that $\mathrm{Li}((Ax)^\beta) - \mathrm{Li}(x^\beta)$ increases as β increases, so we can just let $\beta = 1 - c_4|\Delta_K|^{-1/d}$.

Thus we can deduce from (2.8) and partial integration that, for $A > 1$ and $x > \exp(10d(\log|\Delta_K|)^2)$, we have $\pi_K(Ax) > \pi_K(x)$ if

$$\text{(2.10)}\qquad \frac{\beta Ax - (Ax)^\beta}{\beta \log Ax} + \int_{(Ax)^\beta}^{Ax} \frac{dt}{(\log t)^2} > \frac{\beta x - x^\beta}{\beta \log x} + \int_{x^\beta}^{x} \frac{dt}{(\log t)^2} + 2c_1 Ax \exp\biggl(-c_2\sqrt{\frac{\log x}{d}}\biggr).$$

Taking derivatives, for fixed β we see that $\int_{x^\beta}^{x} dt/(\log t)^2$ is increasing in x for $x > 4$ when $\beta > 1/2$. Thus we can drop the integral terms from both sides of (2.10) to get an inequality that still yields $\pi_K(Ax) > \pi_K(x)$. Rewriting this, we get

$$\text{(2.11)}\qquad \frac{\beta - (Ax)^{\beta-1}}{\beta - x^{\beta-1}}\,\frac{\log x}{\log Ax} > \frac{1}{A} + \frac{2\beta c_1 \log x}{\beta - x^{\beta-1}} \exp\biggl(-c_2\sqrt{\frac{\log x}{d}}\biggr).$$

We see the left hand side is larger than $\log x/\log Ax$ for any $A > 1$, and furthermore $\log x/\log Ax - 1/A > 0$ as x gets large. Thus again we just need to show that there exists c_A, independent of d and $|\Delta_K|$, such that when $x > \exp(c_A d(\log|\Delta_K|)^2)$ we can make the remaining exponential term arbitrarily small. Let $c_A > 10$ and $c_2\sqrt{c_A} > 5/2$. If $x > \exp(c_A d(\log|\Delta_K|)^2)$, we can choose c_A to be large enough independently of d and $|\Delta_K|$ such that

$$\text{(2.12)}\qquad \frac{2\beta c_1 \log x}{\beta - x^{\beta-1}} \exp\biggl(-c_2\sqrt{\frac{\log x}{d}}\biggr) < \frac{2\beta c_1 c_A d(\log|\Delta_K|)^2 |\Delta_K|^{-c_2\sqrt{c_A}}}{\beta - \exp(c_A d(\log|\Delta_K|)^2(\beta-1))}.$$

Letting $\beta = 1 - c_4|\Delta_K|^{-1/d}$, we have

$$\text{(2.13)}\qquad \frac{2\beta c_1 \log x}{\beta - x^{\beta-1}} e^{-c_2\sqrt{(\log x)/d}} < \frac{2c_1 c_A d(\log|\Delta_K|)^2|\Delta_K|^{1/d - c_2\sqrt{c_A}}}{|\Delta_K|^{1/d}\bigl(1 - \exp(-c_4 c_A d(\log|\Delta_K|)^2|\Delta_K|^{-1/d})\bigr) - c_4}.$$

It is not difficult to see that the denominator of this upper bound is bounded below and positive for sufficiently large c_A, independent of $|\Delta_K|$ and d. So in this case we get

$$\text{(2.14)}\qquad \frac{2\beta c_1 \log x}{\beta - x^{\beta-1}} e^{-c_2\sqrt{(\log x)/d}} \ll c_A d(\log|\Delta_K|)^2|\Delta_K|^{1/d - c_2\sqrt{c_A}},$$

where the implied constant can be made independent of c_A, d and $|\Delta_K|$ provided that c_A is sufficiently large. From (2.5) we see that we can make (2.14) independent of $|\Delta_K|$ and d provided $c_2\sqrt{c_A} > 5/2$, and further we see that this bound goes to zero as $c_A \to \infty$.

If the extension is not normal but there exists a tower of fields $\mathbb{Q} \subset K_1 \subset K_2 \subset \cdots \subset K_m = K$ such that K_i is normal over K_{i-1}, then Stark's bound allows for the possibility that

$$1 - c_4|\Delta_K|^{-1/d} < \beta < 1 - (16\log|\Delta_K|)^{-1} \tag{2.15}$$

if $16c_4\log|\Delta_K| > |\Delta_K|^{1/d}$. When this occurs, (2.13) becomes

(2.16)

$$\frac{2\beta c_1\log x}{\beta - x^{\beta-1}}e^{-c_2\sqrt{(\log x)/d}} < \frac{32c_1c_Ad(\log|\Delta_K|)^3|\Delta_K|^{-c_2\sqrt{c_A}}}{16(\log|\Delta_K|)\big(1-\exp\big(-\frac{1}{16}c_Ad(\log|\Delta_K|)\big)\big)-1},$$

and again we see that this bound uniformly goes to zero as $c_A \to \infty$.

Finally, if the extension is not normal, nor does there exist a tower of field extensions as above, then Stark's bound allows for the possibility that

$$1 - c_4|\Delta_K|^{-1/d} < \beta < 1 - (4d!\log|\Delta_K|)^{-1} \tag{2.17}$$

if $4c_4d!\log|\Delta_K| > |\Delta_K|^{1/d}$. When this occurs, (2.13) becomes

$$\begin{aligned}\frac{2\beta c_1\log x}{\beta - x^{\beta-1}}&e^{-c_2\sqrt{(\log x)/d}}\\ &< \frac{8c_1c_Ad(\log|\Delta_K|)^3|\Delta_K|^{-c_2\sqrt{c_A}}d!}{4d!(\log|\Delta_K|)\big(1-\exp\big(-\frac{1}{4}c_Ad(\log|\Delta_K|)/d!\big)\big)-1}.\end{aligned} \tag{2.18}$$

This bound still decays as $c_A \to \infty$, and is uniform in $|\Delta_K|$ but not necessarily in d. We see from (2.5) and Stirling's approximation that if we take $x > \exp(c_Ad(\log d)^2(\log|\Delta_K|)^2)$ instead, effectively replacing each occurrence of c_A in (2.18) with $c_A(\log d)^2$, this bound can be made uniform in d. Indeed, it is enough that we take $x > \exp(c_Ad(\log d)^{2-2\alpha}(\log|\Delta_K|)^2)$ so long as $\log|\Delta_K| \gg d(\log d)^\alpha$. ■

3. Bertrand's postulate from counting ideals

Proof of Theorem 1.5. For an ideal $\mathfrak{a}$ in $\mathcal{O}_K$, let $\Lambda_K(\mathfrak{a}) := \log N(\mathfrak{p})$ when $\mathfrak{a} = \mathfrak{p}^k$ and zero otherwise, where $\mathfrak{p}$ denotes a prime ideal in $\mathcal{O}_K$ above a prime ideal $(p) \subset \mathbb{Z}$. This is the natural extension of the von Mangoldt function to K, where

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_{\mathfrak{a}\subseteq\mathcal{O}_K}\frac{\Lambda_K(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty}\frac{\Lambda_K^{\#}(n)}{n^s}. \tag{3.1}$$

We similarly define the *Chebyshev function* for K,

$$\psi_K(x) := \sum_{N(\mathfrak{a})\le x}\Lambda_K(\mathfrak{a}) = \sum_{n\le x}\Lambda_K^{\#}(n). \tag{3.2}$$

LEMMA 3.1. *For $x \ge 1$,*

$$\sum_{n\le x}\frac{\Lambda_K^{\#}(n)}{n} = \sum_{N(\mathfrak{p})\le x}\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} + O(d), \tag{3.3}$$

where the above right-hand sum is over norms of prime ideals.

For $1 > \alpha > 0$*,*

$$\sum_{n\le x}\frac{\Lambda_K^{\#}(n)}{n^\alpha} \ll d\frac{x^{1-\alpha}-\alpha}{1-\alpha}. \tag{3.4}$$

Proof. Let

$$\phi(x) := \sum_{n\le x}\frac{\Lambda_K^{\#}(n)}{n} - \sum_{N(\mathfrak{p})\le x}\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})}. \tag{3.5}$$

We see that

$$\begin{aligned} |\phi(x)| &\le \sum_{N(\mathfrak{p})\le x}\sum_{n=2}^{\infty}\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^n} \le \sum_{p\le x}\sum_{\mathfrak{p}\cap\mathbb{Z}=(p)}\sum_{n=2}^{\infty}\frac{\log N(\mathfrak{p})}{p^n} \\ &\le \sum_{p\le x}\sum_{n=2}^{\infty}\frac{d\log p}{p^n} \le d\sum_{p\le x}\frac{\log p}{p^2-p} \le d\sum_{m=2}^{\infty}\frac{2\log m}{m^2} = O(d), \end{aligned} \tag{3.6}$$

which gives (3.3). To get (3.4) we can use Abel's partial summation formula to deduce that

$$\sum_{n\le x}\frac{\Lambda_K^{\#}(n)}{n^\alpha} = \psi_K(x)x^{-\alpha} + \alpha\int_1^x \psi_K(u)u^{-\alpha-1}\,du. \tag{3.7}$$

Since $\Lambda_K^{\#}(n) \le d\Lambda(n)$, where $\Lambda(n) := \Lambda_{\mathbb{Q}}(n)$ is the classical von Mangoldt function, we find that $\psi_K(x) \le d\psi(x)$ for all $x \ge 1$, where $\psi(x) := \psi_{\mathbb{Q}}(x)$ is the classical Chebyshev function. It is easily shown that $\psi(x) \ll x$, so $\psi_K(x) \ll dx$. Putting this into (3.7) we get

$$\sum_{n\le x}\frac{\Lambda_K^{\#}(n)}{n^\alpha} \ll d\bigg(x^{1-\alpha} + \frac{\alpha}{1-\alpha}(x^{1-\alpha}-1)\bigg), \tag{3.8}$$

which is a restatement of (3.4). ■

We see that, by the unique prime factorization of ideals in the ring of integers of a number field, we have

$$\begin{aligned} \sum_{n\le x} c_K(n)\log n &= \sum_{N(\mathfrak{a})\le x}\log N(\mathfrak{a}) = \sum_{N(\mathfrak{a})\le x}\sum_{\mathfrak{b}\mid\mathfrak{a}}\Lambda_K(\mathfrak{b}) \\ &= \sum_{N(\mathfrak{b})\le x}\Lambda_K(\mathfrak{b})\sum_{N(\mathfrak{a})\le x/N(\mathfrak{b})} 1 = \sum_{n\le x}\Lambda_K^{\#}(n)\sum_{m\le x/n} c_K(m). \end{aligned} \tag{3.9}$$

Letting

$$f_2(x,K) := \sum_{n\le x} c_K(n)\log n - \rho_K(x\log x - x + 1), \tag{3.10}$$

we see that (3.9) becomes

$$\rho_K(x\log x - x + 1) + f_2(x,K) = \sum_{n\le x} \Lambda_K^{\#}(n)(\rho_K x/n + f_1(x/n,K)). \tag{3.11}$$

Thus by (3.3) we have

$$\sum_{N(\mathfrak{p})\le x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} = \log x + \frac{1}{x\rho_K}\Big(f_2(x,K) - \sum_{n\le x} \Lambda_K^{\#}(n) f_1(x/n,K)\Big) + O(d). \tag{3.12}$$

Now combining (1.9) and (3.4) leads to

$$\sum_{n\le x} \Lambda_K^{\#}(n) f_1(x/n,K) \ll d\mathcal{C}_K \frac{x - \alpha x^{\alpha}}{1-\alpha}. \tag{3.13}$$

To bound $f_2(x,K)$ we make use of the following lemma.

LEMMA 3.2. *If for $0<\alpha<1$ and $x\ge 1$,*

$$|f_1(x,K)| \le \mathcal{C}_K x^{\alpha} \tag{3.14}$$

for some $\mathcal{C}_K > 0$, then

$$|f_2(x,K)| \le \mathcal{C}_K x^{\alpha}\bigg(\log x + \frac{1-x^{-\alpha}}{\alpha}\bigg). \tag{3.15}$$

Proof. This follows from the bound in (1.9) by another application of Abel's summation formula. Indeed,

$$\sum_{n\le x} c_K(n)\log n = \Big(\sum_{n\le x} c_K(n)\Big)\log x - \int_1^x \Big(\sum_{n\le u} c_K(n)\Big)\frac{du}{u} \tag{3.16}$$

so we have

$$\begin{aligned}\sum_{n\le x} c_K(n)\log n &= (\rho_K x + f_1(x,K))\log x - \int_1^x (\rho_K u + f_1(u,K))\frac{du}{u}\\ &= \rho_K(x\log x - x + 1) + f_1(x,K)\log x - \int_1^x f_1(u,K)\frac{du}{u},\end{aligned} \tag{3.17}$$

and (3.15) follows. ∎

Substituting (3.13) and (3.15) back into (3.12) we get

$$\sum_{N(\mathfrak{p})\le x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} = \log x + O\bigg(\frac{\mathcal{C}_K}{\rho_K}\bigg(\frac{d(1-\alpha x^{\alpha-1})}{1-\alpha} + x^{\alpha-1}\log x + \frac{x^{\alpha-1} - x^{-1}}{\alpha}\bigg) + d\bigg). \tag{3.18}$$

And so finally, for any $A \geq 1$,

$$\sum_{N(\mathfrak{p})\leq Ax} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} - \sum_{N(\mathfrak{p})\leq x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} = \log A + O\left(\frac{\mathcal{C}_K}{\rho_K}\frac{d+2}{1-\alpha} + d\right). \tag{3.19}$$

Thus, for any $x > 1$, a prime ideal, $\mathfrak{p}$, exists in $\mathcal{O}_K$ such that $N(\mathfrak{p}) \in [x, Ax]$, so long as

$$\log A \gg \frac{\mathcal{C}_K}{\rho_K}\frac{d+2}{1-\alpha} + d, \tag{3.20}$$

which proves Theorem 1.5. ■

Appendix. Ideal counting. The proof of Proposition 1.9 proceeds with only subtle variation from the proof of the main theorem due to Kuo and Murty in their relevant work [4]. Changes are made to account for the presence of the simple pole at $s = 1$ and that our goal is to minimize growth in the $|\Delta_K|$ aspect, possibly at the expense of growth in the x aspect.

First we use the following result due to Rademacher [8], arising from the sharper version of the Phragmén–Lindelöf theorem, to obtain the convexity bound for $\zeta_K(s)$ in the critical strip.

THEOREM A.1 (Rademacher [8]). *For $\sigma, \eta, t \in \mathbb{R}$ such that $3/2 \geq \sigma > 1$ and $1 - \sigma < \eta < \sigma$, we have*

$$\zeta_K(\eta + it) \leq 3\left(|\Delta_K|\left(\frac{|1+\eta+it|}{2\pi}\right)^d\right)^{(\sigma-\eta)/2}\frac{|1+\eta+it|}{|\eta-1+it|}\zeta_K(\sigma). \tag{A.1}$$

Proof of Proposition 1.9. Henceforth we will use the following notation:

$$\int_{(c,T)} f(s)\,ds := \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s)\,ds. \tag{A.2}$$

Let $T \geq 1$, $2 > c > 1$ and, unless stated otherwise, let $x \in \mathbb{N} + 1/2$. By Perron's formula, we have

$$\begin{aligned}\sum_{n\leq x} c_K(n) \\ = \int_{(c,T)} \zeta_K(s)\frac{x^s}{s}\,ds + O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^c c_K(n)\min\left(1, \frac{1}{T|\log(x/n)|}\right)\right).\end{aligned} \tag{A.3}$$

Thus by Cauchy's residue theorem, for $0 < \eta < 1$,

$$\sum_{n\leq x} c_K(n) = \rho_K x + E_1 + E_2 + E_3 \tag{A.4}$$

where

$$E_1 := \int_{(\eta,T)} \zeta_K(s)\frac{x^s}{s}\,ds \tag{A.5}$$

$$E_2 := \frac{1}{2\pi i}\Bigg(\int_{\eta+iT}^{c+iT} \zeta_K(s)\frac{x^s}{s}\,ds - \int_{\eta-iT}^{c-iT} \zeta_K(s)\frac{x^s}{s}\,ds\Bigg) \tag{A.6}$$

$$E_3 := O\Bigg(\sum_{n=1}^{\infty}\Big(\frac{x}{n}\Big)^c c_K(n)\min\Big(1, \frac{1}{T|\log(x/n)|}\Big)\Bigg). \tag{A.7}$$

We will now use Theorem A.1 to bound $E_1 + E_2 + E_3$. Since $1 + t^2/\eta^2 \gg (1+|t|)^2$, we have

$$\begin{aligned} |E_1| &\ll \Big|\int_{(\eta,T)} \zeta_K(s)\frac{x^s}{s}\,ds\Big| \\ &\ll_d |\varDelta_K|^{(c-\eta)/2}\zeta_K(c)x^\eta \int_{-T}^{T} \frac{(1+|t|)^{d(c-\eta)/2+1}}{|\eta-1+it|\,|\eta+it|}\,dt \\ &\ll_d \frac{1}{\eta(1-\eta)}|\varDelta_K|^{(c-\eta)/2}\zeta_K(c)x^\eta \int_0^T (1+t)^{d(c-\eta)/2-1}\,dt \\ &\ll_d \frac{1}{\eta(1-\eta)^2}|\varDelta_K|^{(c-\eta)/2}\zeta_K(c)x^\eta(1+T)^{d(c-\eta)/2}. \end{aligned} \tag{A.8}$$

Similarly,

$$\begin{aligned} |E_2| &\ll \Bigg(\Big|\int_{\eta+iT}^{c+iT} \zeta_K(s)\frac{x^s}{s}\,ds\Big| + \Big|\int_{\eta-iT}^{c-iT} \zeta_K(s)\frac{x^s}{s}\,ds\Big|\Bigg) \\ &\ll_d \zeta_K(c)(1+T)\int_\eta^c (|\varDelta_K|(1+T)^d)^{(c-\lambda)/2}x^\lambda|\lambda+iT|^{-1}|1-\lambda+iT|^{-1}\,d\lambda. \end{aligned} \tag{A.9}$$

Since for $T \geq 1$ we have $|\lambda + iT|\,|1-\lambda+iT| \gg (1+T)^2$, it follows that

$$\begin{aligned} |E_2| &\ll_d \zeta_K(c)|\varDelta_K|^{c/2}(1+T)^{dc/2-1}\int_\eta^c \Bigg(\frac{x}{\sqrt{|\varDelta_K|}\,(1+T)^{d/2}}\Bigg)^\lambda d\lambda \\ &\ll_d \zeta_K(c)|\varDelta_K|^{c/2}(1+T)^{dc/2-1}\Bigg|\log\frac{x}{\sqrt{|\varDelta_K|}\,(1+T)^{d/2}}\Bigg|^{-1} \\ &\quad\times\Bigg|\Bigg(\frac{x}{\sqrt{|\varDelta_K|}\,(1+T)^{d/2}}\Bigg)^c - \Bigg(\frac{x}{\sqrt{|\varDelta_K|}\,(1+T)^{d/2}}\Bigg)^\eta\Bigg|. \end{aligned} \tag{A.10}$$

So when

$$\Bigg|\log\frac{x}{\sqrt{|\varDelta_K|}\,(1+T)^{d/2}}\Bigg| \geq \log(3/2), \tag{A.11}$$

we have

$$|E_2| \ll_d \zeta_K(c)\left(\frac{x^c}{1+T} + x^\eta|\Delta_K|^{(c-\eta)/2}(1+T)^{d(c-\eta)/2-1}\right). \tag{A.12}$$

When

$$\left|\log \frac{x}{\sqrt{|\Delta_K|}\,(1+T)^{d/2}}\right| \le \log(3/2), \tag{A.13}$$

we note that $(y^c - y^\eta)/\log y$ is bounded uniformly for $y \in [2/3, 3/2]$ and all specified values of c and η. So we can just say $(y^c - y^\eta)/\log y \ll y^c$ in this range. Thus the bound on E_2 given in (A.12) holds regardless of the relationship between x and $\sqrt{|\Delta_K|}\,(1+T)^{d/2}$.

Finally, recalling that $x \in \mathbb{N} + 1/2$ and that $|\log(1-x)| \gg |x|$ for $x \in [-1, 1/2)$, we have

(A.14)

$$\begin{aligned}
E_3 &:= x^c \sum_{n=1}^{\infty} \frac{c_K(n)}{n^c} \min\left(1, \frac{1}{T|\log(x/n)|}\right) \le x^c \sum_{n=1}^{\infty} \frac{c_K(n)}{n^c} \frac{1}{T|\log(x/n)|} \\
&\ll \frac{x^c}{T}\zeta_K(c) + \sum_{x/2<n\le 2x} \left(\frac{x}{n}\right)^c c_K(n) \frac{1}{T|\log(x/n)|} \\
&\ll \frac{x^c}{T}\zeta_K(c) + \frac{1}{T} \sum_{x/2<n\le 2x} c_K(n) \frac{n}{|x-n|} \\
&\ll \frac{x^c}{T}\zeta_K(c) + \frac{C_{d,\varepsilon} x^{1+\varepsilon}}{T} \sum_{x/2\le n\le 2x} \frac{1}{|x-n|}
\end{aligned}$$

for small $\varepsilon > 0$, where $C_{d,\varepsilon}$ is a constant such that

$$c_K(n) \le C_{d,\varepsilon} n^\varepsilon \tag{A.15}$$

for all $n \in \mathbb{N}$. Now since

$$\sum_{x/2\le n\le 2x} \frac{1}{|n-x|} \le 2 \sum_{j=0}^{x-1/2} \frac{1}{j+1/2} \ll \log(1+x), \tag{A.16}$$

if we let $\varepsilon = (c-1)/2$, then for $T \ge 1$ we find that (A.14) becomes

$$|E_3| \ll_\delta \frac{x^c}{1+T}\zeta_K(c) + \frac{C^{(2)}_{d,(c-1)/2} x^{(c+1)/2}}{1+T}. \tag{A.17}$$

where the implicit constant $C_{d,(c-1)/2}$ is changed to $C^{(2)}_{d,(c-1)/2}$ to account for the implied constant in the bound $\log x \ll_\varepsilon x^\varepsilon$. We remark that we do not bother measuring the contribution of the degree d, nor that of $\delta = c-1$, in our main theorem as $C^{(2)}_{d,\delta/2}$ is likely much worse than reality. It is unclear at present how to remove the dependence on this term.

Combining (A.8), (A.12), and (A.17) we get

$$
\begin{aligned}
\text{(A.18)} \quad & |E_1| + |E_2| + |E_3| \\
& \ll_d \zeta_K(c)\left(\frac{x^\eta(|\Delta_K|(1+T)^d)^{(c-\eta)/2}}{\eta(1-\eta)^2} + \frac{x^c}{1+T}\right) + \frac{C^{(2)}_{d,(c-1)/2}x^{(c+1)/2}}{1+T}.
\end{aligned}
$$

For some small $\delta > 0$ such that $1/(3d) > \delta$, let $c = 1+\delta$ and $\eta = 1-\delta$, then let

$$
\text{(A.19)} \qquad 1 + T = x^{2\delta/(1+d\delta)}.
$$

From this, (A.18) becomes

$$
\text{(A.20)} \qquad \sum_{n\le x} c_K(n) = \rho_K x + O_{d,\delta}\big((\zeta_K(1+\delta)|\Delta_K|^\delta + 1)x^{1-\delta/2}\big)
$$

when $x \in \mathbb{N} + 1/2$, and since $T \ge 1$, we also have the constraint $x \ge Q$ due to (A.19), where Q is a constant dependent on δ and d. For $x < Q$ we can just let $T = 1$ in (A.18), and since $x^c \le x^\eta Q^{2\delta}$ in this range we can say (A.20) holds for all $x \in \mathbb{N} + 1/2$ and $x \ge 1$.

For $x \notin \mathbb{N}+1/2$ we can replace x with $\lfloor x\rfloor+1/2$ in $\sum_{n\le x} c_K(n) - \rho(x)$ and note the difference will be on the order of ρ_K at most. Since $\zeta_K(1+\delta) \sim \rho_K/\delta$ we can replace $\zeta_K(1+\delta)$ with $\zeta_K(1+\delta/2)$ to supersede that ρ_K term, and thus get (A.20) for all x and complete the proof of the proposition. ∎

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