TCV Proceedings No. 15, 2011, pp. 89–101.

L-Series and Transcendental Numbers

M. Ram Murty

Department of Mathematics, Queen's University, Kingston, Ontario, K7L 3N6, Canada
e-mail: murty@mast.queensu.ca

Abstract.

Keywords.

In honour of Professor Vasudevan's 60th birthday

1. Introduction

This paper is a survey of some recent work of mine done jointly with V. Kumar Murty, N. Saradha, S. Gun and P. Rath. In all of these works, the motivating question is the following. Given an automorphic representation π, we consider the L-series $L(s, \pi)$ attached to π according to the Langlands formalism. We are interested in the possible transcendence of special values $L(k, \pi)$ when k is a positive integer. Sometimes, we will be interested in algebraic linear combinations of such values. There are many conjectures in the literature regarding the arithmetic significance of $L(k, \pi)$ for certain classes of π, but in this paper, we are concerned with its transcendence.

In these investigations, a central role is played by the celebrated theorem of Alan Baker, proved in 1966. This theorem states the following. If $\alpha_1, \dots, \alpha_n$ are non-zero algebraic numbers and $\beta_1, \dots, \beta_n$ are algebraic numbers, then

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental. The latter case arises if $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$ and $\beta_1, \dots, \beta_n$ are not all zero. The latter case also arises if $\beta_1, \dots, \beta_n$ are linearly independent over $\mathbb{Q}$ (see for example, [20]).

2. Dirichlet's formula

The simplest case of study arises when χ is a non-trivial Dirichlet character mod q and $s = 1$. In this case, one can prove that (see for example, [10])

$$L(1, \chi) = \sum_{a=1}^{q-1} c_a \log(1 - \zeta^a),$$

where the c_a's are algebraic numbers and ζ is a primitive q-th root of unity. Baker's theorem implies that this number is either zero or transcendental. But the well-known theorem of Dirichlet that $L(1, \chi) \neq 0$ now implies that $L(1, \chi)$ is transcendental. This was observed in [1].

3. Class group L-functions

The next "simplest" case to consider is that of Hecke characters of an imaginary quadratic field K. It is convenient to consider a slightly more general context. Let H_K be the ideal class group of K and $f : H_k \to \bar{\mathbb{Q}}$ be an algebraic valued function on H_K not identically zero. As shown in [9], the series

$$\sum_{\mathfrak{a}} \frac{f(\mathfrak{a})}{N(\mathfrak{a})^s}$$

extends to an entire function if and only if

$$\sum_{\mathfrak{a} \in H_K} f(\mathfrak{a}) = 0.$$

In the latter case, we show that $L(1, f)/\pi$ is transcendental. In particular, it is non-zero. This allows us to deduce the following result: the values $L(1, \chi)$ are linearly independent over $\bar{\mathbb{Q}}$ as χ ranges over the characters of H_K. Moreover, all of these numbers are transcendental with at most one exception.

As a consequence of these investigations, V. Kumar Murty and I showed that if χ_D is the quadratic character corresponding to the imaginary quadratic field K, then π and

$$\exp\left(\frac{L'(1, \chi_D)}{L(1, \chi_D)} - \gamma\right),$$

where γ is Euler's constant, are algebraically independent. In particular, the latter exponential is transcendental.

This result raises the interesting question of when we can have $L'(1, \chi) = 0$ for such quadratic characters. For various reasons, to be discussed below, it seems reasonable to conjecture that this never happens for Dirichlet L-series. In this direction, V. Kumar Murty and I proved that for q prime, the number of characters (mod q) for which $L'(1, \chi) = 0$ is $O(q^\epsilon)$ for any $\epsilon > 0$.

The question of non-vanishing of $L'(1, \chi)$ is related to a conjecture due to Lang and Rohrlich. This conjecture predicts that the numbers $\log \Gamma(a/q)$ with $0 < a < q$ and $(a, q) = 1$ are linearly independent over $\bar{\mathbb{Q}}$. Assuming this conjecture, S. Gun, P. Rath and I showed that $L'(1, \chi) \neq 0$ with at most one exceptional character χ mod q.

A weaker conjecture is the assertion that the above numbers are linearly independent over $\mathbb{Q}$. At present, this weaker conjecture is also unknown. However, in the case $q = 3, 4$ and 6, S. Gun, P. Rath and I were able to show that this weaker conjecture is true. This result is an attractive application of a theorem of Chudnovsky (see p. 308 of [6]) and since the proof is short, we give it in the case of $q = 3$ (with the cases $q = 4$ and $q = 6$ being similar).

Suppose that $\log \Gamma(1/3)$ and $\log \Gamma(2/3)$ are linearly dependent over $\mathbb{Q}$. Then,

$$\Gamma(1/3)^a \Gamma(2/3)^b = 1$$

for some integers a, b not both zero. But by the well-known functional equation

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z},$$

we have

$$\Gamma(1/3)\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}.$$

Thus,

$$\Gamma(1/3)^{a-b}(2\pi/\sqrt{3})^b = 1.$$

But this contradicts a 1976 result of Chudnovsky (see p. 308 of [6]) that π and $\Gamma(1/3)$ are algebraically independent.

4. The Chowla and Erdös conjectures

In 1965, Erdös (see [8]) conjectured that if f is a function defined on the residue classes mod q with $f(q) = 0$ and $f(a) = \pm 1$ otherwise, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

It is not hard to show that the conjecture is true if q is even. In recent work with N. Saradha, I was able to show that the conjecture is true if $q \equiv 3 \pmod 4$.

There is a related conjecture of Chowla [4]. In 1969, Chowla asked at the Stony Brook conference if there is a rational valued function, not identically zero, defined on the residue classes mod p such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0.$$

In a beautiful paper by Baker, Birch and Wirsing [2], this question was answered in the negative using Baker's theory of linear forms in logarithms. They studied the case of the general modulus q instead of the prime modulus.

5. Schanuel's conjecture

The celebrated Lindemann-Weierstrass theorem states that if $\alpha_1, \ldots, \alpha_n$ are algebraic numbers which are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent. The conjecture of Schanuel asserts that the algebraicity hypothesis can be dropped. Thus, the conjecture says (as cited in Lang's monograph on Transcendental Numbers) that if $x_1, \ldots, x_n$ are linearly independent over $\mathbb{Q}$, then

$$\text{tr deg}\ \mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n.$$

This is still a major unsolved problem and many interesting consequences emerge from it. For example, we immediately deduce that e and π are algebraically independent. To see this, consider $x_1 = 1, x_2 = \pi i$. These are linearly independent over $\mathbb{Q}$ and so the transcendence degree of

$$\mathbb{Q}(\pi, e) = \mathbb{Q}(1, \pi i, e, e^{\pi i})$$

is at least 2. Thus, e and π are algebraically independent. Consequently, Schanuel implies that both $e+\pi$ and $e\pi$ are transcendental, for otherwise, we have an algebraic relation between them. In particular, 1 and $\log\pi$ are linearly independent over $\mathbb{Q}$ for if not, we have $a + b\log\pi = 0$, for some integers a, b from which we get $e^a\pi^b = 1$, contradicting the algebraic independence of e and π. To deduce that $\log\pi$ is transcendental, we consider $\pi i, \log\pi$ which are linearly independent over $\mathbb{Q}$ since $\pi \neq \pm 1$. Thus, π and $\log\pi$ are algebraically independent. In particular, $\log\pi$ is transcendental. Many interesting results can be deduced from this conjecture.

One particular result that is relevant to our discussion below is a strengthening of Baker's theorem. Recall that Baker's theorem says that if $\log\alpha_1, \ldots, \log\alpha_n$ are logarithms of algebraic numbers linearly independent over $\mathbb{Q}$, then they are linearly independent over $\bar{\mathbb{Q}}$. Schanuel's conjecture predicts the algebraic independence of these numbers. We refer to this as the weak Schanuel conjecture since it plays a major role in our discussion below.

6. Artin L-series

Our motivating question in the context of Artin L-series can be formulated precisely in the following way. If χ is an Artin character, which does not contain the trvial character, then is it true that $L(1, \chi)$ is transcendental? It seems reasonable to conjecture that the answer is yes, thus giving a non-abelian generalization of the result for Dirichlet characters that was discussed at the outset.

In this context, Stark [19] has made specific conjectures about the special value $L(1, \chi)$ as the determinant of a regulator-like matrix involving units of number fields. If χ is a rational character, Stark was able to prove his conjecture.

In joint work with S. Gun and P. Rath, I [13] was able to show that if we admit Stark's conjecture, along with the weak Schanuel conjecture, then one can deduce that $L(1, \chi)$ is transcendental. If χ is a rational character, we can drop the assumption of Stark's conjecture since that is known. These are first steps towards resolving the general conjecture that $L(1, \chi)$ is transcendental for any Artin character χ.

7. Catalan's constant

The study of $L(s, \chi)$ at $s = 1$ suggests an investigation of other values like $L(k, \chi)$ where k is a positive integer greater than 1. Even in the case of classical Dirichlet characters, many things are still shrouded in mystery and inevitably lead to questions as difficult as the transcendence of $\zeta(3)$.

If χ is a Dirichlet character (mod q), so that χ and k have the same parity, then one can show, following Euler, that $L(k, \chi)$ is a non-zero algebraic multiple of π^k. Consequently, $L(k, \chi)$ is transcendental in these cases. If k and χ have opposite parity, then the arithmetic study of $L(k, \chi)$ seems impenetrable at the moment.

The "simplest" case arises if $q = 4$ and χ is the (odd) non-trivial character mod 4. In this case, the number $L(2, \chi)$ is called Catalan's constant and it is unknown at present if it is irrational. In 2003, Rivoal and Zudilin [17] showed that at least one of the seven numbers, $L(2k, \chi)$, with $k = 1, 2, \ldots, 7$, is irrational.

Part of the difficulty is exposed when one attempts at an explicit evaluation of $L(2, \chi)$. This involves Barnes double gamma function so that the transcendence of Catalan's constant is intimately tied up with the study of special values of the classical gamma function and the Barnes double gamma function.

In 1900, Barnes [3] introduced the double gamma function $G(z)$. It has a Weierstrass factorization:

$$G(z+1) = (2\pi)^z e^{-z(z+1)/2+\gamma z^2/2} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^n e^{-z+z^2/(2n)}.$$

One can show that

$$2L(2,\chi)/\pi = 3 \log \Gamma(1/4) - \log \Gamma(3/4) 4 \log G(1/4) - 4 \log G(3/4).$$

S. Gun, P. Rath and myself [14] were able to show that at least one of the numbers

$$L(2,\chi)/\pi^2, \quad G(1/4)/G(3/4),$$

is transcendental. Presumably, both numbers are transcendental.

8. Multiple gamma functions and $\zeta(3)$

A similar result can be proved for $\zeta(3)/\pi^3$ by relating it to multiple gamma functions. These functions were also introduced by Barnes and are defined inductively as follows. Let $\Gamma_1(z)$ be the classical gamma function. The multiple gamma functions Γ_n satisfy the functional equation

$$\Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)},$$

with the initial condition $\Gamma_n(1) = 1$. $\Gamma_2(z)$ is Barnes double gamma function.

With S. Gun and P. Rath, I [14] proved that at least one of

$$\zeta(3)/\pi^3, \quad \Gamma_2(1/2)\Gamma_3(1/2),$$

is transcendental.

9. Chowla's problem revisited

As mentioned earlier, in 1969, Chowla asked if there is a non-zero rational valued function f with prime period such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0.$$

In 1973, Baker, Birch and Wirsing [2] answered Chowla's question by proving the following. Let f be an algebraic valued function, not identically zero, defined on the integers with period q (not necessarily prime). Suppose that $f(r) = 0$ for $1 < (r, q) < q$ and the q-th cyclotomic polynomial is irreducible over the field $\mathbb{Q}(f(n) : n \in \mathbb{Z})$. Then,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

The proof of this theorem uses Baker's theory of linear forms in logarithms.

In the setting of Chowla's problem, the series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

can be written as a linear form in logarithms. Consequently, for any f satisfying the conditions of the theorem of Baker, Birch and Wirsing, we deduce that the series is transcendental.

In the special case f is a Dirichlet character χ, we combine the classical theorem of Dirichlet that $L(1, \chi)$ in not zero along with Baker's theorem to deduce that it is transcendental. This was observed in [1]. If $(q, \varphi(q)) = 1$, then the Baker–Birch–Wirsing theorem allows us to deduce that the $L(1, \chi)$ as χ ranges over the non-trivial characters (mod q) are linearly independent over the rationals.

As noted in [10], these results allow us to deduce something new about the digamma function. Recall that the digamma function, $\psi(x)$ is the logarithmic derivative of the gamma function. Its connection to Chowla's problem is revealed via Hurwitz's zeta function:

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s}.$$

This series admits an analytic continuation to the entire complex plane apart from a simple pole at $s = 1$ with residue 1. In fact, one can show that

$$\zeta(s, x) = \frac{1}{s - 1} - \psi(x) + O(s - 1),$$

as $s \to 1$.

The analytic continuation of $L(s, f)$ can be determined from the analytic continuation of the Hurwitz zeta function. More precisely,

$$L(s, f) = q^{-s} \sum_{a=1}^{q} f(a)\zeta(s, a/q).$$

Using this, one can show that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

converges if and only if

$$\sum_{a=1}^{q} f(a) = 0.$$

In the case of convergence, we have

$$L(1, f) = -\frac{1}{q}\sum_{a=1}^{q} f(a)\psi(a/q). \tag{1}$$

If ζ denotes a primitive q-th root of unity, we can write

$$f(n) = \sum_{a=1}^{q} \hat{f}(a)\zeta^{an}.$$

Here

$$\hat{f}(a) = \frac{1}{q}\sum_{b=1}^{q} f(b)\zeta^{-ab}.$$

Using this, it is not hard to see that

$$L(1, f) = -\sum_{a=1}^{q-1} \hat{f}(a)\log(1-\zeta^{a}).$$

Clearly, if f is algebraic valued, so is $\hat{f}$. Consequently, the right hand side is by the theorem of Baker, zero or transcendental. Thus, if $L(1, f) \neq 0$, then it is transcendental.

The above connection with the Hurwitz zeta function allows us to deduce some new results about the digamma function. For instance, we can show that there is at most one algebraic number in the list $\psi(a/q)$ with $0 < a < q$ and $(a, q) = 1$. Indeed, if both $\psi(a_1/q)$ and $\psi(a_2/q)$ are algebraic, then choosing $f(a_1) = 1$, $f(a_2) = -1$ and $f(a) = 0$ for all the other classes, we see that the conditions of the Baker, Birch and Wirsing theorem are satisfied. So $L(1, f) \neq 0$ and transcendental. On the other hand, from formula (1), we see that $L(1, f)$ is algebraic. This is a contradiction.

10. Euler constants

In 1975, Lehmer [7] defined generalized Euler constants as follows:

$$\gamma(a, q) = \lim_{x\to\infty}\left(\sum_{n\le x, n\equiv a(\bmod\ q)} \frac{1}{n} - \frac{\log x}{q}\right).$$

In the case $q = 1$, this is Euler's constant γ. One expects that all of these generalized Euler constants are transcendental. This is unknown at the moment. N. Saradha and I [11] showed that in the infinite list

$$\gamma, \quad \gamma(a, q), \quad 1 \le a < q, \quad q \ge 2,$$

there is at most one algebraic number.

11. The Chowla–Milnor conjecture

So far, we have discussed certain L-series at $s = 1$. Now we shall discuss L-series at other positive integer arguments. In 1982, P. and S. Chowla [5] made a conjecture about $L(s, f)$ when f is a rational-valued function defined on the residue classes mod q. When q is prime, they predicted that $L(2, f) \neq 0$ unless $f(1) = f(2) = \cdots = f(p-1) = f(p)/(1-p^2)$. This conjecture is equivalent to the following: the numbers

$$\zeta(2, a/p), \quad 1 \leq a \leq p-1$$

are linearly independent over $\mathbb{Q}$. Inspired by this conjecture, Milnor [15] suggested the following generalization; the numbers

$$\zeta(k, a/q), \quad 1 \leq a \leq q, \quad (a, q) = 1,$$

are linearly independent over $\mathbb{Q}$. We refer to this assertion as the Chowla–Milnor conjecture.

This innocuous looking conjecture has serious implications. In joint work with S. Gun and P. Rath, I [14] showed the following. The Chowla–Milnor conjecture for $q = 4$ is equivalent to the irrationality of

$$\zeta(2k+1)/\pi^{2k+1}$$

for all $k \geq 1$. Unconditionally, we can show that the Chowla–Milnor conjecture is true for k odd and $q = 3$ or $q = 4$.

Our investigations led us to formulate the strong Chowla–Milnor conjecture. This is the following assertion. For any k and $q > 1$, the numbers 1 and $\zeta(k, a/q)$, $1 \leq a < q$ with $(a, q) = 1$ are linearly independent over $\mathbb{Q}$. In joint work with S. Gun and P. Rath, I [14] showed that for k odd, the strong Chowla conjecture is true for either $q = 3$ or for $q = 4$ if and only if $\zeta(k)$ is irrational.

12. The polylog conjecture

The polylogarithm function $Li_k(z)$ is defined for $|z| < 1$ by the series

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

For $k > 1$, we have convergence for $|z| \leq 1$. Let us note that $Li_1(z) = -\log(1-z)$. It seems reasonable to conjecture the following. Suppose that $\alpha_1, \ldots, \alpha_n$ are algebraic numbers such that $Li_k(\alpha_1), \ldots, Li_k(\alpha_n)$ are linearly independent over $\mathbb{Q}$, then they are linearly independent over $\bar{\mathbb{Q}}$. In the case $k = 1$, this is a consequence of the celebrated theorem of Alan Baker.

With S. Gun and P. Rath, I [14] proved that the polylog conjecture implies the Chowla–Milnor conjecture for all k and all q.

13. The Chowla–Milnor space

Let $V_k(q)$ be the $\mathbb{Q}$-vector space spanned by the numbers $\zeta(k, a/q)$ with $1 \leq a < q$, and $(a, q) = 1$. The Chowla–Milnor conjecture is equivalent to the assertion that $\dim V_k(q) = \varphi(q)$.

S. Gun, P. Rath and I [14] were able to show the following. Let $k > 1$ and $q > 1$. Then,

$$\dim_{\mathbb{Q}} V_k(q) \geq \varphi(q)/2.$$

If k is odd and r is coprime to q, then either

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2} + 1,$$

or

$$\dim_{\mathbb{Q}} V_k(r) \geq \frac{\varphi(r)}{2} + 1.$$

In particular, there exists a q_0 such that

$$\dim_{\mathbb{Q}} V_k(q) \geq \frac{\varphi(q)}{2} + 1,$$

for any q coprime to q_0.

14. Multiple zeta values

There is a surprising connection between the theory of multiple zeta values and the Chowla–Milnor conjecture that emerged in our researches. To explain this, we begin with some definitions.

The multiple zeta function is defined by

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}.$$

We usually consider this function with the s_i's positive integers and with $s_1 > 1$ for convergence reasons. Let V_k be the $\mathbb{Q}$-vector space spanned by the values $\zeta(s_1, \ldots, s_t)$ with $t \geq 1$, $s_1 > 1$ and $s_1 + \cdots + s_t = k$. Let d_k be the dimension of V_k.

Zagier [21] has conjectured that d_k satisfies the Fibonacci-like recurrence

$$d_k = d_{k-1} + d_{k-3}.$$

The recurrence is easily analysed with initial conditions, $d_0 = 1$, $d_1 = 0$ and $d_2 = 1$. It is apparent that d_k has exponential growth.

If we define $\delta_0 = 1$, $\delta_1 = 0$, $\delta_2 = 1$ and

$$\delta_k = \delta_{k-2} + \delta_{k-3},$$

then Deligne, Goncharov and Terasoma (independently) showed that $d_k \leq \delta_k$ for every value of k. However, to date, no example of any value of k is known for which $d_k \geq 2$.

It is not hard to see that $d_2 = d_3 = d_4 = 1$. It is also not difficult to show that $d_5 = 2$ if and only if $\zeta(2)\zeta(3)/\zeta(5)$ is irrational. Similarly, one can show that $d_6 = 2$ if and only if $\zeta^2(3)/\pi^6$ is irrational.

In joint work with S. Gun and P. Rath, I [14] was able to show that if the Chowla–Milnor conjecture is true for all values of q and k, then

$$\left(\frac{\zeta(2k+1)}{\pi^{2k+1}}\right)^2$$

is irrational and $d_{4k+2} \geq 2$ for all $k \geq 1$.

In 1948, Petersson [16] proved that $7\zeta(3)/\pi^3 = (f, g)_\Gamma$ where Γ is the theta subgroup of $SL_2(\mathbb{Z})$. This raises the interesting question of transcendental values of the Petersson inner product.

In joint work with S. Gun and P. Rath, I [13] was able to show the following. Let f be a non-zero normalized Hecke eigenform of weight 1 and level N. Suppose that f has rational Fourier coeffficients. Then, assuming the weak Schanuel conjecture, the inner product (f, f) is transcendental.

15. Kronecker's limit formula

Let K be an imaginary quadratic field and H_K its ideal class group. For each ideal class C of H_K, we have the ideal class zeta function, $\zeta(s, C)$. Hecke showed how to analytically continue this zeta function to the entire complex plane, apart from a simple pole at $s = 1$. Kronecker's celebrated limit formula (see for example, [18]) states that

$$\lim_{s\to 1} \zeta(s, C) = \frac{2\pi}{w\sqrt{|d_K|}}\left(\frac{1}{s-1} + 2\gamma - \log|d_K| - \frac{1}{6}\log g(C^{-1})\right) + O(s-1),$$

where d_K is the discriminant of K and w is the number of roots of unity in K. Here g is defined as follows. If $\mathfrak{a}$ is an ideal with basis u, v where $\Im(u/v) > 0$, we define

$$g(\mathfrak{a}) = (2\pi)^{-12} N(\mathfrak{a})^6 |\Delta(u/v)|,$$

where $\Delta(z)$ is Ramanujan's cusp form.

One can formulate a K-analogue of Chowla's problem. Let $f : H_K \to \bar{\mathbb{Q}}$ be not identically zero. Set

$$L(s, f) = \sum_{\mathfrak{a}} \frac{f(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

Using the theory of the ideal class zeta function, Kumar Murty and I [9] showed that $L(s, f)$ converges at $s = 1$ if and only if

$$\sum_{C\in H_K} f(C) = 0.$$

In the latter case, one can prove that $L(1, f) \neq 0$ and

$$-\frac{L(1, f)}{\pi} = \frac{1}{3w\sqrt{|d_K|}} \sum_{C\in H_K} f(C)\log g(C^{-1}).$$

This formula is not in a suitable form to apply Baker's theorem. To put it in an amenable form, we apply the theory of complex multiplication. This theory says that if C_1 and C_2 are two ideal classes, then $g(C_1)/g(C_2)$ is an algebraic number lying in the Hilbert class field of K. Since

$$\sum_{C\in H_K} f(C) = 0,$$

we can re-write our expression as

$$-\frac{L(1,f)}{\pi} = \frac{1}{3w\sqrt{|d_K|}} \sum_{C\in H_K} f(C) \log g(C^{-1})/g(C_0).$$

The right hand side is now a $\bar{\mathbb{Q}}$-linear combination of logarithms of algebraic numbers. Since $L(1, f) \neq 0$, we deduce from Baker's theorem that $L(1, f)/\pi$ is transcendental.

16. The Chowla–Selberg formula

Applying the previous discussion to the case of an ideal class character χ, we deduce a formula for $L(1,\chi)$ and infer that $L(1,\chi)/\pi$ is transcendental.

In the particular case that χ is a genus character, we can deduce more. Indeed, if χ_D is the character corresponding to the imaginary quadratic field K, we deduce

$$\frac{L'(1,\chi_D)}{L(1,\chi_D)} = \gamma - \log |d_K| - \frac{1}{6h} \sum_{C\in H_K} \log g(C),$$

where h denotes the order of H_K.

This allows us to give a new proof of the Chowla–Selberg formula. Indeed, by the functional equation, we can relate

$$\frac{L'(1,\chi_D)}{L(1,\chi_D)} \quad \text{with} \quad \frac{L'(0,\chi_D)}{L(0,\chi_D)}.$$

On the other hand,

$$L(s,\chi_D) = D^{-s} \sum_{a=1}^{D} \chi_D(a)\zeta(s, a/D).$$

Applying the formula of Lerch (see for example, [15])

$$\zeta'(0,x) = \log(\Gamma(x)/\sqrt{2\pi}),$$

as well as the class number formula for $L(0,\chi_D)$, we deduce the celebrated Chowla–Selberg formula.

Putting everything together, we get

$$\prod_{C\in H_K} g(C)^6 = \left(\frac{1}{4\pi\,|d_K|}\right)^h \prod_{a=1}^{D} \Gamma(a/D)^{w\chi_D(a)/2}.$$

By the work of Deligne-Gross, the left hand side is (up to an algebraic factor) the period of an elliptic curve with CM by the ring of integers of K. In other words,

$$\prod_{a=1}^{D} \Gamma(a/D)^{\chi_D(a)h},$$

is a product of a power of π and a power of a period of a CM elliptic curve (up to an algebraic factor).

17. Transcendental values of the Γ-function

The preceding analysis can be used to obtain new transcendental values of the Γ-function. If q is a divisor of 24, it is not hard to see that all the characters (mod q) are real-valued. Thus, if $q|24$, then $\Gamma(a/q)$ can be expressed as a product of periods of non-isogenous CM elliptic curves and π. Following Grothendieck, it seems reasonable to conjecture that the fundamental periods of non-isogenous CM elliptic curves and π are algebraically independent. If we admit this conjecture, then we can deduce that $\Gamma(a/q)$ with $q|24$ is transcendental. In particular, $\Gamma(1/8)$ will be transcendental.

It is possible to deduce something unconditionally. We can show that at most one of $\Gamma(1/8), \Gamma(3/8), \Gamma(5/8), \Gamma(7/8)$ is algebraic. Here is a sketch of the proof. Suppose that at least two of these numbers are algebraic. By the Chowla–Selberg formula, we get an algebraic dependency relation between π and a period of a CM elliptic curve. This contradicts a theorem of Chudnovsky that states that these numbers are algebraically independent.

Acknowledgements

I am grateful to Drs. Sanoli Gun and Purusottam Rath for their corrections of an earlier version of this paper. I also thank the referee for helpful remarks.

References

[1] S. Adhikari, N. Saradha, T. N. Shorey and R. Tijdeman, Transcendental infinite sums, *Indag. Math. (N.S.)*, **12** (2001) no. 1, 1–14.

[2] A. Baker, B. Birch and E. A. Wirsing, On a problem of Chowla, *Journal of Number Theory*, **5** (1973) 224–236.

[3] E. W. Barnes, The theory of the G-function, *Quarterly Journal of Pure and Applied Math.*, **31** (1900) 264–314.

[4] S. Chowla, The non-existence of non-trivial relations between roots of a certain irreducible equation, *Journal of Number Theory*, **2** (1970) 120–123.

[5] P. Chowla and S. Chowla, On irrational numbers, *K. Norske Vidensk. Selsk. Skr.*, **3** (1982) 1–5. (See also: The Collected Papers of Sarvadaman Chowla, Volume 3, 1962–1986, edited by J. G. Huard and K. S. Williams, Centre de Recherches Mathématiques, 1999, pp. 1383–1387.)

[6] G. V. Chudnovsky, Contributions to the theory of transcendental numbers, Mathematical Surveys and Monographs, Number 19, Amer. Math. Society, Providence, Rhode Island (1984).

[7] D. H. Lehmer, Euler constants for arithmetical progressions, *Acta Arithmetica*, **27** (1975) 125–142.

[8] A. E. Livingston, The series $\sum_1^\infty f(n)/n$ for periodic f, *Canadian Math. Bulletin*, **8** (1965) 413–432.

[9] M. Ram Murty and V. Kumar Murty, Transcendental values of class group L-functions, *Mathematische Annalen*, published online December 12 (2010).

[10] M. Ram Murty and N. Saradha, Transcendental values of the digamma function, *Journal of Number Theory*, **125** (2007) 298–318.

[11] M. Ram Murty and N. Saradha, Euler-Lehmer constants and a conjecture of Erdös, *Journal of Number Theory*, **130** (2010) no. 12, 2671–2682.

[12] M. Ram Murty and N. Saradha, Special values of the polygamma functions, *International Journal of Number Theory*, **5** (2009) no. 2, 257–270.

[13] S. Gun, M. Ram Murty and P. Rath, Transcendental values of special values of L-series, *Canadian Journal of Mathematics*, published online November 6 (2010).

[14] S. Gun, M. Ram Murty and P. Rath, On a conjecture of Chowla and Milnor, to appear in *Canadian Journal of Mathematics*.

[15] J. Milnor, On polylogarithms, Hurwitz zeta functions and the Kubert identities, *Enseign. Math.*, **29(2)** (1983) no. 3–4, 281–322.

[16] H. Petersson, Über die Berechnung der Skalarprodukte ganzer Modulformen, *Comm. Math. Helvetici*, **22** (1983) no. 1, 168–199.

[17] T. Rivoal and W. Zudilin, Diophantine properties of numbers related to Catalan's constant, *Math. Annalen*, **326** (2003) 705–721.

[18] C. L. Siegel, Advanced analytic number theory, TIFR Publications, Narosa Press.

[19] H. Stark, L-functions at $s = 1$, II, Artin L-functions with rational characters, *Advances in Math.*, **17** (1975) no. 1, 60–92.

[20] R. Tijdeman, Hilbert's seventh problem: on the Gel'fond-Baker method and its applications, in Mathematical developments arising from Hilbert problems, *Proc. Symp. Pure Math.* (1974) 241–268, Amer. Math. Society, Providence, Rhode Island (1976).

[21] D. Zagier, Values of zeta functions and their applications, First European Congress of Mathematics, Vol. 2 (Paris, 1992), Progress in Mathematics, **120**, Birkhäuser (1994).